

The Complete One-Loop Dilation Operator of $\mathcal{N} = 2$ SuperConformal QCD

Pedro Liendo^{a*}, Elli Pomoni^{b†} and Leonardo Rastelli^{a‡}

^a *C.N. Yang Institute for Theoretical Physics,
Stony Brook University,
Stony Brook, NY 11794-3840, USA*

^b *Institut für Mathematik und Institut für Physik,
Humboldt-Universität zu Berlin
Johann von Neumann-Haus, Rudower Chaussee 25, 12489 Berlin, Germany*

ABSTRACT:

We evaluate the full planar one-loop dilation operator of $\mathcal{N} = 2$ SuperConformal QCD, the $SU(N_c)$ super Yang-Mills theory with $N_f = 2N_c$ fundamental hypermultiplets, in the flavor-singlet sector. Remarkably, the spin-chain Hamiltonian turns out to be completely fixed by superconformal symmetry, as in $\mathcal{N} = 4$ SYM. We present a more general calculation, for the superconformal quiver theory with $SU(N_c) \times SU(N_c)$ gauge group, which interpolates between $\mathcal{N} = 2$ SCQCD and the \mathbb{Z}_2 orbifold of $\mathcal{N} = 4$ SYM; here symmetry fixes the Hamiltonian up to a single parameter, corresponding to the ratio of the two marginal gauge couplings.

KEYWORDS: AdS/CFT, Integrability.

*Email: pedro.liendo@stonybrook.edu

†Email: pomoni@mathematik.hu-berlin.de

‡Email: leonardo.rastelli@stonybrook.edu

Contents

1. Introduction	2
2. Preliminaries	4
2.1 Field Content and Symmetries	4
2.2 The Spin Chain	5
3. Lifting the Full One-loop Hamiltonian from a Subsector	6
3.1 Superconformal Projectors	7
3.2 A Convenient Subsector	8
4. Field Theory Evaluation of the Hamiltonian	9
4.1 $\mathcal{V} \times \mathcal{V}$	9
4.2 $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$	11
4.3 $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$	13
4.3.1 $SU(2)_L$ singlet	15
4.3.2 $SU(2)_L$ triplet	16
5. Algebraic Evaluation of the Hamiltonian	16
5.1 First order expressions for $\mathcal{Q}(g)$ and $\mathcal{S}(g)$	17
5.2 $\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$	19
5.3 $\bar{\mathcal{V}} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \bar{\mathcal{V}}$	20
5.4 $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$	21
6. The Harmonic Action	23
6.1 $\mathcal{V} \times \mathcal{V}$	23
6.2 $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$	23
6.3 $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$	25
6.3.1 $SU(2)_L$ singlet	25
6.3.2 $SU(2)_L$ triplet	26
7. Discussion	26
A. $\mathcal{N} = 2$ Superconformal Multiplets	27
B. Oscillator Representation	28
B.1 Vector multiplets \mathcal{V} and $\bar{\mathcal{V}}$	29
B.2 Hypermultiplet \mathcal{H}	29
B.3 Two-letter Superconformal Primaries	29

C. A Sample Field Theory Calculation	31
D. Two Closed Subsectors and the Magnon S-matrix	33
D.1 S-matrix in the Right Sector	35
D.1.1 $\bar{\psi} \bar{\psi}$ and $\bar{\psi} \bar{\tilde{\psi}}$ scattering	35
D.1.2 $\bar{\psi} Q$, $Q \bar{\psi}$, $\bar{Q} \bar{\psi}$ and $\bar{\tilde{\psi}} \bar{Q}$ scattering	35
D.2 S-matrix in the Left Sector.	36

1. Introduction

Perturbative field theory calculations of the dilation operator have played a crucial role in uncovering the integrability properties of $\mathcal{N} = 4$ super Yang-Mills (SYM) (see *e.g.* [1, 2, 3, 4, 5, 6] for a partial list of references and [7] for a recent comprehensive review). As the integrability structure is common to the planar field theory and the dual string sigma model, one might even imagine an alternative history where the AdS/CFT correspondence is discovered following the hints of the field theory integrability.

In this paper we present a calculation of the complete planar one-loop dilation operator of a paradigmatic $\mathcal{N} = 2$ superconformal theory, the $SU(N_c)$ super Yang-Mills theory with $N_f = 2N_c$ fundamental hypermultiplets, in the flavor singlet sector. This theory is perhaps the simplest 4d conformal field theory outside the “universality class” of $\mathcal{N} = 4$ SYM and is a very interesting case study. It admits a large N expansion in the Veneziano sense of $N_f \sim N_c \rightarrow \infty$ with $\lambda = g_{YM}^2 N_c$ fixed, and a perturbative expansion in the exactly marginal ’t Hooft coupling λ . Is the planar theory integrable? Does it have a dual string description? Some progress in answering these two questions, which are logically independent, was described in [8, 9]. In particular in [9] the planar one-loop dilation operator in the scalar subsector was obtained, with some tantalizing hints of integrability. As explained in [8, 9], it is illuminating to embed $\mathcal{N} = 2$ superconformal QCD (SCQCD) into the $\mathcal{N} = 2$ $SU(N_c) \times SU(N_{\tilde{c}})$ quiver theory (with $N_{\tilde{c}} \equiv N_c$) which has two independent marginal couplings g_{YM} and \check{g}_{YM} . The quiver theory interpolates between the standard \mathbb{Z}_2 orbifold of $\mathcal{N} = 4$ SYM for $\check{g}_{YM} = g_{YM}$ and SCQCD for $\check{g}_{YM} = 0$. With minor extra work, we can keep the calculations in this paper more general and derive the full one-loop spin chain Hamiltonian for the whole interpolating quiver theory. In the closed subsector of scalar chiral fields the Hamiltonian of the quiver theory has been very recently obtained to three loops [10].

The quiver theory is known to be integrable at the orbifold point $\check{g}_{YM} = g_{YM}$ [11], but it is definitely not integrable for generic values of the couplings, since the two-body magnon S-matrix does not obey the Yang-Baxter equation [9]. It is still an open question whether integrability is recovered in the (somewhat singular) SCQCD limit $\check{g}_{YM} \rightarrow 0$. We expect the evaluation of the full one-loop dilation operator presented here to be a crucial step towards answering this question.

We find that the full spin-chain Hamiltonian of $\mathcal{N} = 2$ SCQCD is completely fixed by symmetry, as is the case for $\mathcal{N} = 4$ SYM. This came to us as a surprise, because representation theory is less restrictive for the $\mathcal{N} = 2$ superconformal algebra. Unlike $\mathcal{N} = 4$ SYM, where each site of the spin chain hosts a single ultrashort irreducible representation, in our case single-site letters decompose into three distinct irreps, and the tensor product of two nearest-neighbor state spaces has a considerably more intricate decomposition. Nevertheless, by a non-trivial generalization of Beisert’s approach [12, 13], we find that symmetry is sufficient to determine the Hamiltonian up to overall normalization. We regard this as a hint to deeper solvability/integrability properties than meet the eye. The generalization to the interpolating quiver is then as simple as one may hope: symmetry leaves a single undetermined parameter, which gets identified with the ratio of the two marginal gauge couplings.

After reviewing some basics and setting notations in Section 2, we describe the strategy of our calculation in Section 3. Following Beisert [12, 13], the evaluation of the full one-loop dilation operator consists of two parts. First, one computes the dilation operator in a closed subsector with $SU(1, 1)$ symmetry; then one uses superconformal symmetry to uplift the result to the full theory. The details are considerably more involved than in the $\mathcal{N} = 4$ case: the two-site state space is spanned by a baroque list of irreducible representations, some of which appear in different copies, leading to an intricate mixing problem. Nevertheless, we are able to identify a suitable subsector, whose Hamiltonian uplifts to the full theory. The complete Hamiltonian is written as a sum of two-site superconformal projectors.

We compute the Hamiltonian in the closed subsector both by direct evaluation of Feynman diagrams (Section 4) and by a purely algebraic approach using the constraints of superconformal symmetry (Section 5). The algebraic method is similar to the one used by Beisert in his thesis [13] for $\mathcal{N} = 4$ SYM, and rather surprisingly leads to a similar uniqueness result. In both cases the key feature is the existence of a centrally-extended $SU(1|1)$ symmetry, which commutes with the bosonic $SU(1, 1)$ symmetry up to local gauge transformations on the spin chain. Finally in Section 6 we re-write the Hamiltonian, so far expressed rather implicitly as a sum over superconformal projectors, in the much more explicit “harmonic action” [12] form, which is easy to implement on any given state. Algebraic techniques to obtain spin-chain Hamiltonians were also used in [14, 15] for $\mathcal{N} = 4$ SYM at higher loops and in [16] for the ABJM theory.

The interpolating quiver theory, while not integrable, is interesting in its own right. It has a dual string description as the IIB background $AdS_5 \times S^5/\mathbb{Z}_2$, with a non-trivial period of B_{NSNS} through the collapsed cycle of the orbifold [17, 18]. For generic values of the couplings the symmetry of the spin chain in the excitation picture contains a single copy of the supergroup $SU(2|2)$ (as opposed to the two independent $SU(2|2)$ s of the $\mathcal{N} = 4$ chain). The two-body S-matrix of magnons transforming under the surviving $SU(2|2)$ can be determined to all orders in the gauge coupling [4, 19], up to an overall phase ambiguity, from symmetry considerations alone. Armed with the explicit one-loop Hamiltonian, in Appendix D we confirm the prediction of [19] to lowest order in the coupling. Three other technical appendices complement the text.

	$SU(N_c)$	$U(N_f)$	$SU(2)_R$	$U(1)_r$
$\mathcal{Q}_\alpha^{\mathcal{I}}$	1	1	2	$+1/2$
$\mathcal{S}_{\mathcal{I}}^\alpha$	1	1	2	$-1/2$
A_μ	Adj	1	1	0
ϕ	Adj	1	1	-1
$\lambda_{\mathcal{I}\alpha}$	Adj	1	2	$-1/2$
$Q_{\mathcal{I}}$	\square	\square	2	0
ψ_α	\square	\square	1	$+1/2$
$\tilde{\psi}_\alpha$	$\overline{\square}$	$\overline{\square}$	1	$+1/2$

Table 1: Field content and symmetries of $\mathcal{N} = 2$ SCQCD. We show the quantum numbers of the Poincaré supercharges $\mathcal{Q}_\alpha^{\mathcal{I}}$, of the conformal supercharges $\mathcal{S}_{\mathcal{I}}^\alpha$ and of the elementary component fields. Conjugate objects (such as $\tilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$ and $\tilde{\phi}$) are not written explicitly.

2. Preliminaries

We begin by quickly reviewing $\mathcal{N} = 2$ superconformal QCD, the closely related \mathbb{Z}_2 quiver theory, and the structure of their spin chains. For more details, including the explicit Lagrangians, we refer to [9].

2.1 Field Content and Symmetries

We summarize in Table 1 the field content and quantum numbers of the $\mathcal{N} = 2$ SYM theory with gauge group $SU(N_c)$ and $N_f = 2N_c$ fundamental hypermultiplets, which we refer to as $\mathcal{N} = 2$ superconformal QCD (SCQCD). Its global symmetry group is $U(N_f) \times SU(2)_R \times U(1)_r$, where $SU(2)_R \times U(1)_r$ is the R-symmetry subgroup of the superconformal group. We use indices $\alpha, \beta = \pm$ and $\dot{\alpha}, \dot{\beta} = \pm$ for the Lorentz group, $\mathcal{I}, \mathcal{J} = 1, 2$ for $SU(2)_R$, $i, j = 1, \dots, N_f$ for the flavor group $U(N_f)$ and $a, b = 1, \dots, N_c$ for the color group $SU(N_c)$. The $\mathcal{N} = 2$ vector multiplet consists of a gauge field A_μ , two Weyl spinors $\lambda_{\mathcal{I}\alpha}$, $\mathcal{I} = 1, 2$, which form a doublet under $SU(2)_R$, and one complex scalar ϕ , all in the adjoint representation of $SU(N_c)$. Each $\mathcal{N} = 2$ hypermultiplet consists of an $SU(2)_R$ doublet $Q_{\mathcal{I}}$ of complex scalars and of two Weyl spinors ψ_α and $\tilde{\psi}_\alpha$, $SU(2)_R$ singlets.

$\mathcal{N} = 2$ SCQCD, which has one exactly marginal coupling g_{YM} , can be viewed as a limit of the $\mathcal{N} = 2$ \mathbb{Z}_2 quiver theory with gauge group¹ $SU(N_c) \times SU(N_{\check{c}})$, which has two exactly marginal couplings g_{YM} and \check{g}_{YM} , as $\check{g}_{YM} \rightarrow 0$. When $g_{YM} = \check{g}_{YM}$ the quiver theory is the familiar \mathbb{Z}_2 orbifold of $\mathcal{N} = 4$ SYM. Table 2 summarizes the field content and symmetries of the quiver theory. Besides the R-symmetry group $SU(2)_R \times U(1)_r$, the theory has an additional $SU(2)_L$ global symmetry, whose indices we denote by $\hat{\mathcal{I}}, \hat{\mathcal{J}} = \hat{1}, \hat{2}$. Supersymmetry organizes the component fields into the $\mathcal{N} = 2$ vector multiplets of each factor of the gauge

¹The gauge groups are identical, $N_{\check{c}} \equiv N_c$, but we find it useful to distinguish with a “check” all the quantities pertaining to the second gauge group.

	$SU(N_c)$	$SU(N_{\tilde{c}})$	$SU(2)_R$	$SU(2)_L$	$U(1)_R$
$\mathcal{Q}_\alpha^{\mathcal{I}}$	1	1	2	1	$+1/2$
$\mathcal{S}_\mathcal{I}^\alpha$	1	1	2	1	$-1/2$
A_μ	Adj	1	1	1	0
\check{A}_μ	1	Adj	1	1	0
ϕ	Adj	1	1	1	-1
$\check{\phi}$	1	Adj	1	1	-1
$\lambda_{\mathcal{I}\alpha}$	Adj	1	2	1	$-1/2$
$\check{\lambda}_{\mathcal{I}\alpha}$	1	Adj	2	1	$-1/2$
$Q_{\mathcal{I}\hat{\mathcal{I}}}$	\square	$\overline{\square}$	2	2	0
$\psi_{\hat{\mathcal{I}}\alpha}$	\square	$\overline{\square}$	1	2	$+1/2$
$\tilde{\psi}_{\hat{\mathcal{I}}\alpha}$	$\overline{\square}$	\square	1	2	$+1/2$

Table 2: Field content and symmetries of the quiver theory that interpolates between the \mathbb{Z}_2 orbifold of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD.

group, $(\phi, \lambda_{\mathcal{I}}, A_\mu)$ and $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_\mu)$, and into two bifundamental hypermultiplets, $(Q_{\mathcal{I},\hat{1}}, \psi_{\hat{1}}, \tilde{\psi}_{\hat{1}})$ and $(Q_{\mathcal{I},\hat{2}}, \psi_{\hat{2}}, \tilde{\psi}_{\hat{2}})$.

Setting $\check{g}_{YM} = 0$, the second vector multiplet $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_\mu)$ becomes free and completely decouples from the rest of the theory, which coincides with $\mathcal{N} = 2$ SCQCD (the field content is the same and $\mathcal{N} = 2$ susy does the rest). The $SU(N_{\tilde{c}})$ symmetry can now be interpreted as a global flavor symmetry. In fact there is a symmetry enhancement $SU(N_{\tilde{c}}) \times SU(2)_L \rightarrow U(N_f = 2N_c)$: the $SU(N_{\tilde{c}})$ index \check{a} and the $SU(2)_L$ index $\hat{\mathcal{I}}$ can be combined into a single flavor index $i \equiv (\check{a}, \hat{\mathcal{I}}) = 1, \dots, 2N_c$.

We work in the large $N_c \equiv N_{\tilde{c}}$ limit, keeping fixed the 't Hooft couplings

$$\lambda \equiv g_{YM}^2 N_c \equiv 8\pi^2 g^2, \quad \check{\lambda} \equiv \check{g}_{YM}^2 N_{\tilde{c}} \equiv 8\pi^2 \check{g}^2. \quad (2.1)$$

We will often refer to the theory with arbitrary g and \check{g} as the “interpolating SCFT”, thinking of keeping g fixed as we vary \check{g} from $\check{g} = g$ (orbifold theory) to $\check{g} = 0$ ($\mathcal{N} = 2$ SCQCD \oplus extra $N_{\tilde{c}}^2 - 1$ free vector multiplets).

2.2 The Spin Chain

As familiar, the planar dilation operator of a gauge theory can be represented as the Hamiltonian of a spin chain. Each site of the chain is occupied by a “letter” of the gauge theory: a letter $\mathcal{D}^k \mathcal{A}$ can be any of the elementary fields \mathcal{A} acted on by an arbitrary number of gauge-covariant derivatives \mathcal{D} . A closed chain corresponds to a single-trace operator.

In the interpolating SCFT, letters belonging to the vector multiplets are in the adjoint representation of either gauge group (index structures $^a{}_b$ and $^{\check{a}}{}_{\check{b}}$), while letters belonging to

the hypermultiplets are in a bifundamental representation (index structures ${}^a_{\bar{b}}$ and ${}^{\bar{a}}_b$). In SCQCD, vector letters have index structure a_b , while hyper letters have structures a_i and i_b . We restrict attention to the flavor-singlet sector of SCQCD. Then, as explained in [8, 9], in the Veneziano limit of $N_c \rightarrow \infty$, $N_f \rightarrow \infty$ with $N_f/N_c \equiv 2$ and $g_{YM}^2 N_c$ fixed, the basic building blocks are the “generalized single-trace operators”, where consecutive letters have contracted color or flavor indices, for example

$$\text{Tr}[\bar{\phi}\phi\phi Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\bar{\phi}] = \bar{\phi}^a_b \phi^b_c \phi^c_d Q_{\mathcal{I}}^d \bar{Q}^{\mathcal{J}i}_e \bar{\phi}^e_a, \quad a, b, c, d, e = 1, \dots, N_c, \quad i = 1, \dots, N_f. \quad (2.2)$$

In the large N Veneziano limit the action of the dilation operator is well-defined on generalized single-traces, because mixing with multi-traces is suppressed. We write the planar dilation operator as

$$D = g^2 H, \quad (2.3)$$

where H is the spin-chain Hamiltonian. At one-loop, H is of nearest-neighbor form,

$$H = \sum_{\ell=1}^L H_{\ell, \ell+1}. \quad (2.4)$$

The one-loop Hamiltonian of the interpolating theory depends on the ratio of the couplings, $\kappa \equiv \check{g}/g$, while the one-loop Hamiltonian of SCQCD has no parameters. We can obtain H_{SCQCD} as the $\kappa \rightarrow 0$ limit of the interpolating Hamiltonian, restricted to the $U(N_f)$ singlet subsector (consecutive $SU(2)_L$ indices are contracted).

3. Lifting the Full One-loop Hamiltonian from a Subsector

Computing the complete one-loop Hamiltonian appears to be a daunting combinatorial task, because of the sheer number of possible two-letter structures on which the Hamiltonian can act. For $\mathcal{N} = 4$ SYM, Beisert [12] was able to determine the full one-loop Hamiltonian by making maximal use of the power of superconformal symmetry. The letters of $\mathcal{N} = 4$ SYM belong to a single representation of the superconformal algebra, the ultrashort “singleton” representation V_F . The tensor product of two singletons has a simple decomposition into an infinite sum of irreducible representations,

$$V_F \times V_F = \sum_{j=0}^{\infty} V_j. \quad (3.1)$$

The one-loop Hamiltonian can then be written as

$$H_{12} = \sum_{j=0}^{\infty} f(j) P_j, \quad (3.2)$$

where P_j is a projector on the V_j module for letters at sites 1 and 2. Beisert’s strategy was to

identify a simple closed subsector of the theory, such that each of the V_j modules contains a representative within the subsector. The coefficients $f(j)$ and thus the full Hamiltonian can be read off from the Hamiltonian of the closed subsector. A particularly clever choice [13] of subsector is the $SU(1,1) \times U(1|1)$ subsector comprising the letters $D_{++}^n \lambda_+$, where λ_α is one of the four Weyl fermions. The algebraic constraints of superconformal symmetry are so powerful that they fix the Hamiltonian of this sector, up to the overall normalization which corresponds to a rescaling of the coupling.² All in all, the one-loop Hamiltonian of $\mathcal{N} = 4$ SYM is determined by superconformal symmetry alone.

In adapting Beisert's strategy to our case, we are faced with the complication that the letters belong to three distinct representations of the $\mathcal{N} = 2$ superconformal algebra, with their tensor products containing different copies of the same module. This leads to a rather intricate mixing problem. Nevertheless, the problem turns out to be tractable. We are able to identify a subsector from which the full Hamiltonian can be lifted. We have determined the Hamiltonian within the subsector both by explicit Feynman diagram calculations, as described in Section 4, and by exploiting the constraints of the superconformal algebra, as described in Section 5.

3.1 Superconformal Projectors

Our notations for superconformal representations are borrowed from [20] and reviewed in Appendix A. The letters of SCQCD (as well as of the whole interpolating theory) belong to three superconformal representations, which we denote by \mathcal{H} , \mathcal{V} and $\bar{\mathcal{V}}$. The hypermultiplet letters ($Q_{\mathcal{I}}$ and its descendants³) belong to the representation $\mathcal{H} \equiv \hat{\mathcal{B}}_{\frac{1}{2}}$, while the vector multiplet letters split into the two conjugate representations $\mathcal{V} \equiv \bar{\mathcal{E}}_{1(0,0)}$ (ϕ and its descendants) and $\bar{\mathcal{V}} \equiv \mathcal{E}_{1(0,0)}$ ($\bar{\phi}$ and its descendants). It is not difficult, using $\mathcal{N} = 2$ superconformal characters⁴, to evaluate the relevant tensor products⁵

$$\mathcal{H} \times \mathcal{H} = \sum_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})}, \quad (3.3)$$

$$\mathcal{H} \times \mathcal{V} = \sum_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q+1}{2}, \frac{q}{2})} = \mathcal{V} \times \mathcal{H}, \quad (3.4)$$

$$\mathcal{H} \times \bar{\mathcal{V}} = \sum_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q+1}{2})} = \bar{\mathcal{V}} \times \mathcal{H}, \quad (3.5)$$

²In his first calculation [12], Beisert considered the $SU(1,1)$ subsector consisting of the letters $\mathcal{D}_{++}^n Z$, where Z is a complex scalar, and determined the $SU(1,1)$ one-loop Hamiltonian by direct evaluation of Feynman diagrams.

³We are suppressing for now $SU(2)_L$ indices, since $SU(2)_L$ commutes with the superconformal algebra.

⁴See for example [21] for an illustration of superconformal character techniques in $\mathcal{N} = 4$ case.

⁵Following [20], we extend the definition of the $\hat{\mathcal{C}}$ multiplets to $j_1, j_2 = -\frac{1}{2}$ according to the rules: $\hat{\mathcal{C}}_{0(-\frac{1}{2}, -\frac{1}{2})} \equiv \hat{\mathcal{B}}_1$, $\hat{\mathcal{C}}_{0(0, -\frac{1}{2})} \equiv \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$, $\hat{\mathcal{C}}_{0(-\frac{1}{2}, 0)} \equiv \mathcal{D}_{\frac{1}{2}(0,0)}$, $\hat{\mathcal{C}}_{0(\frac{1}{2}, -\frac{1}{2})} \equiv \bar{\mathcal{D}}_{\frac{1}{2}(\frac{1}{2}, 0)}$ and $\hat{\mathcal{C}}_{0(-\frac{1}{2}, \frac{1}{2})} \equiv \mathcal{D}_{\frac{1}{2}(0, \frac{1}{2})}$.

$$\mathcal{V} \times \mathcal{V} = \bar{\mathcal{E}}_{2(0,0)} + \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q+1}{2}, \frac{q-1}{2})}, \quad (3.6)$$

$$\bar{\mathcal{V}} \times \bar{\mathcal{V}} = \mathcal{E}_{2(0,0)} + \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q-1}{2}, \frac{q+1}{2})}, \quad (3.7)$$

$$\mathcal{V} \times \bar{\mathcal{V}} = \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})} = \bar{\mathcal{V}} \times \mathcal{V}. \quad (3.8)$$

The two-site Hamiltonian H_{12} can still be written as a sum of superconformal projectors, but we must take into account mixing between different sectors. For example, since the representation $\hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})}$ appears in the tensor products $\mathcal{H} \times \mathcal{H}$, $\mathcal{V} \times \bar{\mathcal{V}}$ and $\bar{\mathcal{V}} \times \mathcal{V}$, these states will mix. The restriction of H_{12} to this subspace takes the form

$$H_{12} = A_{11}(-1) \mathcal{P}_{(-\frac{1}{2}, -\frac{1}{2})} + \sum_{q=0}^{\infty} \begin{pmatrix} A_{11}(q) & A_{12}(q) & A_{13}(q) \\ A_{21}(q) & A_{22}(q) & A_{23}(q) \\ A_{31}(q) & A_{32}(q) & A_{33}(q) \end{pmatrix} \mathcal{P}_{(\frac{q}{2}, \frac{q}{2})}, \quad (3.9)$$

where for each q the 3×3 matrix $A_{rs}(q)$ is the mixing matrix of $\mathcal{H} \times \mathcal{H}$, $\mathcal{V} \times \bar{\mathcal{V}}$ and $\bar{\mathcal{V}} \times \mathcal{V}$. Similarly, there is mixing between $\mathcal{H} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{H}$, and between $\mathcal{H} \times \bar{\mathcal{V}}$ and $\bar{\mathcal{V}} \times \mathcal{H}$, but no mixing for either $\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$, since these latter products decompose into representations that do not appear anywhere else.

3.2 A Convenient Subsector

A straightforward way to obtain the coefficients that multiply the superconformal projectors would be to evaluate the dilation operator on the superconformal primaries of each module. The projectors act trivially on these states and the mixing matrix could be read immediately. However, the primaries are complicated objects (see Appendix B.3) and it will be easier to consider certain descendants instead.

We have identified a closed subsector, somewhat analogous to the $SU(1,1) \times U(1|1)$ subsector [13] of $\mathcal{N} = 4$ SYM. In SCQCD, our subsector consists of the letters λ_{2+} , $\bar{\lambda}_{2+}$, Q_2 and \bar{Q}_2 , acted upon by an arbitrary number of covariant derivatives \mathcal{D}_{++} . Note that all the $SU(2)_R$ indices are taken to be subscripts⁶ with the value $\mathcal{I} = 2$. In the interpolating theory, we add $\check{\lambda}_{2+}$ and $\check{\bar{\lambda}}_{2+}$ to the list. It will be convenient to define (with $\mathcal{D} \equiv \mathcal{D}_{++}$)

$$\lambda_k = \frac{\mathcal{D}^k}{k!} \lambda_{2+}, \quad \bar{\lambda}_k = \frac{\mathcal{D}^k}{k!} \bar{\lambda}_{2+}, \quad (3.10)$$

$$\check{\lambda}_k = \frac{\mathcal{D}^k}{k!} \check{\lambda}_{2+}, \quad \check{\bar{\lambda}}_k = \frac{\mathcal{D}^k}{k!} \check{\bar{\lambda}}_{2+}, \quad (3.11)$$

$$Q_{k\hat{\mathcal{I}}} = \frac{\mathcal{D}^k}{k!} Q_{2\hat{\mathcal{I}}}, \quad \bar{Q}_{k\hat{\mathcal{I}}} = \frac{\mathcal{D}^k}{k!} \bar{Q}_{2\hat{\mathcal{I}}}. \quad (3.12)$$

⁶If the natural position of the $SU(2)_R$ index is as a superscript, as in $\bar{\lambda}_{\alpha}^{\mathcal{I}}$ and $\bar{Q}^{\mathcal{I}}$, we lower it using $\epsilon_{\mathcal{I}\mathcal{J}}$.

The $SU(2)_L$ indices $\hat{\mathcal{I}} = \hat{1}, \hat{2}$ will often be suppressed to avoid cluttering.

The sector (3.10)-(3.12) is closed to all loops, as one easily checks by using conservation of the engineering dimension and of the Lorentz and the R-symmetry quantum numbers. The subgroup of the superconformal group acting on the sector is $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$. The $SU(1,1)$ generators are

$$\mathcal{J}'_+(g) = \mathcal{P}_{++}(g), \quad (3.13)$$

$$\mathcal{J}'_-(g) = \mathcal{K}^{++}(g), \quad (3.14)$$

$$\mathcal{J}'_3(g) = \frac{1}{2}D_0 + \frac{1}{2}\delta D(g) + \frac{1}{2}\mathcal{L}_+^{++} + \frac{1}{2}\dot{\mathcal{L}}_+^{++}, \quad (3.15)$$

where $\delta D(g) \equiv D(g) - D_0$ is the difference between the quantum dilation operator and its classical limit $D_0 = D(0)$. The states $Q_{k=0}$ and $\bar{Q}_{k=0}$ are primaries of spin $-\frac{1}{2}$ representations of $SU(1,1)$, while the states $\lambda_{k=0}$, $\check{\lambda}_{k=0}$, $\bar{\lambda}_{k=0}$, $\bar{\check{\lambda}}_{k=0}$ are primaries of spin -1 representations of $SU(1,1)$. The $SU(1|1) \times SU(1|1) \times U(1)$ generators will be presented in Section 5, they play a key role in the algebraic approach but will not be important for the analysis of the next Section.

Each of the modules appearing on the right hand side of the tensor products (3.3)-(3.8) contains a representative in this subsector. The representatives are primaries of $SU(1,1)$, and descendants with respect to the full $SU(2,2|2)$. This is sufficient to uplift the Hamiltonian of the subsector to the full Hamiltonian.

4. Field Theory Evaluation of the Hamiltonian

In this section we describe the field-theory evaluation of the one-loop Hamiltonian in the $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$ subsector, and its uplifting to the full Hamiltonian. We present the result for the interpolating theory, as a function of $\kappa = \check{g}/g$. The result for SCQCD is obtained by taking the limit $\kappa \rightarrow 0$ and focussing on the relevant subspace (that is, discarding the “checked” fields and contracting adjacent $SU(2)_L$ indices). We can focus on evaluating the Hamiltonian on two-site states with open indices a_b and $^a_{\check{b}}$, since the Hamiltonian acting on the structures $^{\check{a}}_{\check{b}}$ and $^{\check{a}}_b$ is immediately obtained by interchanging $g \leftrightarrow \check{g}$.

4.1 $\mathcal{V} \times \mathcal{V}$

The states of the $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$ subsector belonging to $\mathcal{V} \times \mathcal{V}$ have the form $\lambda_k \lambda_{n-k}$. The relevant Feynman diagrams are shown in Figures 1 and 2. All our calculations are done in Feynman gauge where the gauge propagator reads $\frac{g_{\mu\nu}}{k^2}$. A sample field theory calculation is described in Appendix C.

The action of the Hamiltonian on these states is

$$H'_{12} \lambda_k \lambda_{n-k} = 2 \sum_{k'=0}^n c_{n,k,k'} \lambda_{k'} \lambda_{n-k'}, \quad (4.1)$$

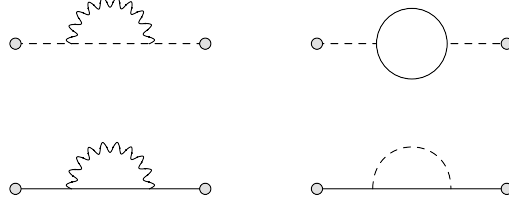


Figure 1: Self-energy corrections of the external legs. Full lines denote fermion propagators, curly lines gauge boson propagators and dashed lines scalar propagators. For $\lambda_k \lambda_{n-k}$ mixing only the second line of diagrams contributes. Corrections to bosonic legs depicted in the first line will be relevant in the next two subsections.

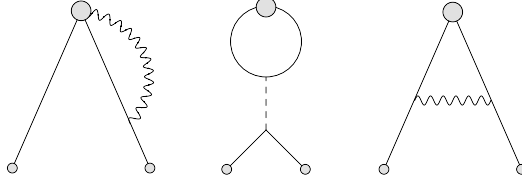


Figure 2: One-loop 1PI Feynman diagrams contributing to $\lambda_k \lambda_{n-k}$ mixing. The last two diagrams are in fact zero for this combination of Lorentz and $SU(2)_R$ indices.

with

$$c_{n,k,k'} = \delta_{k=k'} (h(k+1) + h(n-k+1)) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k > k'}}{n-k'+1} + \frac{\delta_{k < k'}}{k'+1}, \quad (4.2)$$

where $h(k)$ are the harmonic numbers, $h(k) = \sum_{j=1}^k \frac{1}{j}$ and $h(0) \equiv 0$.

Using the oscillator representation (see Appendix B) it is easy to check that H'_{12} is invariant under $SU(1,1)$. We can then write the Hamiltonian density as

$$H'_{12} = \sum_{j=0}^{\infty} A(j) \mathcal{P}'_{-1-j}, \quad (4.3)$$

where \mathcal{P}'_{-1-j} is a projector on the $SU(1,1)$ module of spin $-1-j$. To obtain the coefficients $A(j)$ we act on the $SU(1,1)$ highest weights,

$$\mathcal{J}(j) = -\frac{(j+2)}{(j+1)} \sum_{k=0}^j \frac{(-1)^k}{k+1} \binom{j}{k} \binom{j+1}{k} \mathcal{D}^{j-k} \lambda_{2+} \mathcal{D}^k \lambda_{2+}. \quad (4.4)$$

The result is

$$H'_{12} \mathcal{J}(j) = 4h(j+1) \mathcal{J}(j), \quad (4.5)$$

which implies $A(j) = 4h(j+1)$. The lifting procedure is now straightforward: $\mathcal{J}(j)$ is not only an $SU(1,1)$ highest weight but also a superconformal descendant, it can be obtained by applying $-\frac{1}{2} \mathcal{R}_2^1 \mathcal{Q}_+^2$ to (B.32) for $j=0$ and $\mathcal{Q}_+^1 \tilde{\mathcal{Q}}_{+2}$ to (B.33) for $j>0$. The $SU(1,1)$

modules are sub-modules of the the superconformal modules with $j = q$. The only module not present in this sub-sector is $\bar{\mathcal{E}}_{2(0,0)}$, but we know that this is a protected multiplet so its coefficient is just zero. All in all, the Hamiltonian density in $\mathcal{V} \times \mathcal{V}$ is

$$H_{12} = 0 \times \mathcal{P}_{\bar{\mathcal{E}}} + \sum_{q=0}^{\infty} 4h(q+1) \mathcal{P}_{(\frac{q+1}{2}, \frac{q-1}{2})}. \quad (4.6)$$

4.2 $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$

The mixing between $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \bar{\mathcal{V}}$ should be identical, we only need to focus on the first case. The relevant states are $\lambda_k Q_{n-k} \in \mathcal{V} \times \mathcal{H}$ and $Q_k \check{\lambda}_{n-k} \in \mathcal{H} \times \mathcal{V}$.

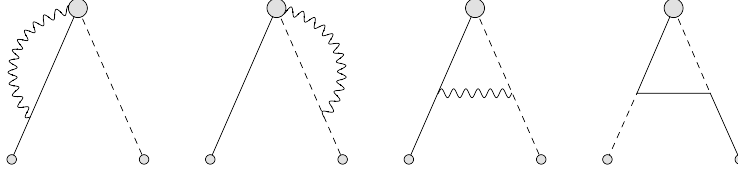


Figure 3: One-loop 1PI Feynman diagrams contributing to $\lambda_k Q_{n-k} \leftrightarrow Q_k \check{\lambda}_{n-k}$ mixing.

The action of the Hamiltonian is

$$H'_{12} \lambda_k Q_{n-k} = 2 \sum_{k'=0}^n a_{n,k,k'} \lambda_{k'} Q_{n-k'} + 2 \sum_{k'=0}^n b_{n,k,k'} Q_{k'} \check{\lambda}_{n-k'}, \quad (4.7)$$

$$H'_{12} Q_k \check{\lambda}_{n-k} = 2 \sum_{k'=0}^n \check{a}_{n,k,k'} Q_{k'} \check{\lambda}_{n-k'} + 2 \sum_{k'=0}^n \check{b}_{n,k,k'} \lambda_{k'} Q_{n-k'}, \quad (4.8)$$

where

$$a_{n,k,k'} = \delta_{k=k'} \left(h(k+1) - \frac{1}{2(n-k+1)} \right) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k < k'}}{k'+1} + \delta_{k=k'} \frac{1+\kappa^2}{4} (h(n-k) + h(n-k+1)), \quad (4.9)$$

$$b_{n,k,k'} = -\kappa \frac{\delta_{k \geq k'}}{n-k'+1}, \quad (4.10)$$

$$\check{a}_{n,k,k'} = \kappa^2 \delta_{k=k'} \left(h(n-k+1) - \frac{1}{2(k+1)} \right) - \kappa^2 \frac{\delta_{k \neq k'}}{|k-k'|} + \kappa^2 \frac{\delta_{k > k'}}{n-k'+1} + \delta_{k=k'} \frac{1+\kappa^2}{4} (h(k) + h(k+1)), \quad (4.11)$$

$$\check{b}_{n,k,k'} = -\kappa \frac{\delta_{k \leq k'}}{k'+1}. \quad (4.12)$$

In this case, the Hamiltonian density H'_{12} is *not* an $SU(1,1)$ invariant. However, conformal symmetry only dictates that the total Hamiltonian $\sum_{\ell} H'_{\ell, \ell+1}$ acting on a *closed* spin chain

must be invariant. A redefinition of the two-site Hamiltonian of the form

$$H'_{\ell,\ell+1} \rightarrow H'_{\ell,\ell+1} - K_\ell + K_{\ell+1}, \quad (4.13)$$

where K_ℓ is a local operator at site ℓ , leaves the total Hamiltonian invariant. So what we must really check is whether we can make the two-site Hamiltonian invariant by an appropriate choice of K_ℓ . The choice of K_ℓ that makes H'_{12} invariant for the whole $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$ subsector is

$$K_\ell = \sum_{k=0}^{\infty} \left(f(k) P_{Q_k}^\ell - f(k) P_{\check{Q}_k}^\ell \right), \quad f(k) = \frac{1-\kappa^2}{2} (h(k) + h(k+1)), \quad (4.14)$$

where $P_{Q_k}^\ell$ is the projector on the state Q_k at site ℓ , and similarly for $P_{\check{Q}_k}^\ell$. We have verified this claim for the restriction of H'_{12} to each of the tensor products. For the tensor products $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$, the transformation (4.13, 4.14) amounts to redefining the coefficients (4.9, 4.11) as

$$a_{n,k,k'} \rightarrow a_{n,k,k'} + \frac{1}{2} f(n-k), \quad \check{a}_{n,k,k'} \rightarrow \check{a}_{n,k,k'} - \frac{1}{2} f(k). \quad (4.15)$$

The new coefficients read

$$a_{n,k,k'} = \delta_{k=k'} (h(k+1) + h(n-k)) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k < k'}}{k'+1}, \quad (4.16)$$

$$\check{a}_{n,k,k'} = \kappa^2 \left(\delta_{k=k'} (h(k) + h(n-k+1)) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k > k'}}{n-k'+1} \right), \quad (4.17)$$

and these combinations *are* $SU(1,1)$ invariant as can be easily checked with the oscillator representation. (The coefficients $b_{n,k,k'}$ and $\check{b}_{n,k,k'}$ were never problematic). Now we can write H'_{12} in (4.7, 4.8) as a sum of projectors

$$H'_{12} = \sum_{j=0}^{\infty} \begin{pmatrix} A_{11}(j) & A_{12}(j) \\ A_{21}(j) & A_{22}(j) \end{pmatrix} \mathcal{P}'_{-\frac{3}{2}-j}. \quad (4.18)$$

To obtain the undetermined coefficients we act on the $SU(1,1)$ highest weights (of spin $-\frac{3}{2}-j$),

$$\mathcal{J}(j) = \sum_{k=0}^j (-1)^k \binom{j}{k} \binom{j+1}{k} \mathcal{D}^{j-k} \lambda_{2+} \mathcal{D}^k Q_2, \quad (4.19)$$

$$\mathcal{K}(j) = \sum_{k=0}^j (-1)^k \binom{j}{k} \binom{j+1}{k+1} \mathcal{D}^{j-k} Q_2 \mathcal{D}^k \check{\lambda}_{2+}. \quad (4.20)$$

As before, these are also superconformal descendants. They can be obtained by applying

$-\frac{1}{2}\mathcal{R}_2^1\mathcal{R}_2^1\mathcal{Q}_+^2$ to (B.36) and (B.39) for $j = 0$ and $\mathcal{Q}_+^1\tilde{\mathcal{Q}}_{+2}$ to (B.37) and (B.40) for $j > 0$.

$$H'_{12}\mathcal{J}(j) = 2(h(j+1) + h(j))\mathcal{J}(j) - \frac{2\kappa}{j+1}\mathcal{K}(j), \quad (4.21)$$

$$H'_{12}\mathcal{K}(j) = 2\kappa^2(h(j+1) + h(j))\mathcal{K}(j) - \frac{2\kappa}{j+1}\mathcal{J}(j). \quad (4.22)$$

The lifting procedure works as before: there is a one-to-one relationship between $SU(1,1)$ modules and superconformal modules, now with $q+1 = j$. The full one-loop result for $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$ is then

$$H_{12} = 2 \sum_{q=-1}^{\infty} \begin{pmatrix} h(q+2) + h(q+1) & -\frac{\kappa}{q+2} \\ -\frac{\kappa}{q+2} & \kappa^2(h(q+2) + h(q+1)) \end{pmatrix} \mathcal{P}_{(\frac{q+1}{2}, \frac{q}{2})}. \quad (4.23)$$

A quick check: Let's consider the action of the Hamiltonian on the two dimensional vector space formed by ϕQ and $Q\check{\phi}$. These are the superconformal primaries of the $q = -1$ modules. The mixing matrix is just (4.23) evaluated at $q = -1$. The result is

$$H_{12} = \begin{pmatrix} 2 & -2\kappa \\ -2\kappa & 2\kappa^2 \end{pmatrix}, \quad (4.24)$$

in perfect agreement with [9]. This is a nice check because in the above calculation we never considered ϕ and $\check{\phi}$.

4.3 $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$

The relevant states are $Q_k\bar{Q}_{n-k}$, $\lambda_k\bar{\lambda}_{n-k}$ and $\bar{\lambda}_k\lambda_{n-k}$.

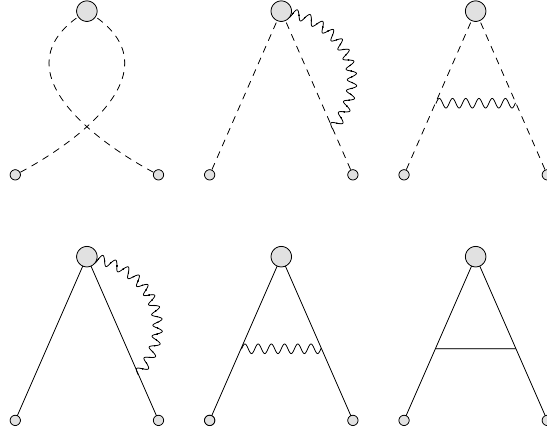


Figure 4: The diagrams in the first row contribute to $Q_k\bar{Q}_{n-k}$ mixing, the diagrams in the second row to $\lambda_k\bar{\lambda}_{n-k}$ and $\bar{\lambda}_k\lambda_{n-k}$ mixing.

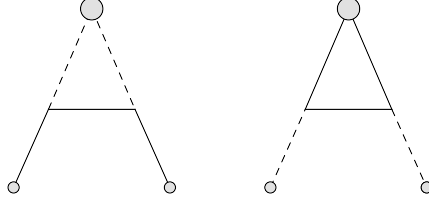


Figure 5: 1PI diagrams contributing to the mixing $Q_k \bar{Q}_{n-k} \leftrightarrow (\lambda_k \bar{\lambda}_{n-k}, \bar{\lambda}_k \lambda_{n-k})$

The action of the Hamiltonian on the squarks, *after* the redefinition (4.13, 4.14) to make it $SU(1,1)$ invariant, is

$$H'_{12} Q_k \hat{\mathcal{Q}}_{n-k}^{\hat{\mathcal{J}}} = 2 \sum_{k'=0}^n (a_{n,k,k'})_{\hat{\mathcal{I}}\hat{\mathcal{L}}}^{\hat{\mathcal{J}}\hat{\mathcal{K}}} Q_{k'}^{\hat{\mathcal{K}}} \bar{Q}_{n-k'}^{\hat{\mathcal{L}}} + 2\delta_{\hat{\mathcal{I}}}^{\hat{\mathcal{J}}} \sum_{k'=0}^{n-1} (b_{n,k,k'} \lambda_{k'} \bar{\lambda}_{n-k'-1} + c_{n,k,k'} \bar{\lambda}_{k'} \lambda_{n-k'-1}) , \quad (4.25)$$

where (with $\hat{\mathbb{I}} \equiv \delta_{\hat{\mathcal{I}}}^{\hat{\mathcal{K}}} \delta_{\hat{\mathcal{L}}}^{\hat{\mathcal{J}}}$, $\hat{\mathbb{K}} \equiv \delta_{\hat{\mathcal{I}}}^{\hat{\mathcal{J}}} \delta_{\hat{\mathcal{L}}}^{\hat{\mathcal{K}}}$)

$$a_{n,k,k'} = \frac{\hat{\mathbb{K}}}{(n+1)} + \kappa^2 \hat{\mathbb{I}} \left(\delta_{k=k'} (h(k) + h(n-k)) - \frac{\delta_{k \neq k'}}{|k - k'|} \right) , \quad (4.26)$$

$$b_{n,k,k'} = \frac{1}{n+1} \left(-\frac{\delta_{k > k'}}{n - k'} + \frac{\delta_{k \leq k'}}{k' + 1} \right) , \quad (4.27)$$

$$c_{n,k,k'} = -\frac{1}{n+1} \left(-\frac{\delta_{k > k'}}{n - k'} + \frac{\delta_{k \leq k'}}{k' + 1} \right) . \quad (4.28)$$

For the action on the fermions, we get

$$H_{12} \lambda_k \bar{\lambda}_{n-k} = 2 \sum_{k'=0}^n (a_{n,k,k'} \lambda_{k'} \bar{\lambda}_{n-k'} + b_{n,k,k'} \bar{\lambda}_{k'} \lambda_{n-k'}) + 2 \sum_{k'=0}^{n+1} c_{n,k,k'} Q_{k'} \hat{\mathcal{Q}}_{n+1-k'}^{\hat{\mathcal{I}}} , \quad (4.29)$$

$$H_{12} \bar{\lambda}_k \lambda_{n-k} = 2 \sum_{k'=0}^n (a_{n,k,k'} \bar{\lambda}_{k'} \lambda_{n-k'} + b_{n,k,k'} \lambda_{k'} \bar{\lambda}_{n-k'}) - 2 \sum_{k'=0}^{n+1} c_{n,k,k'} Q_{k'} \hat{\mathcal{Q}}_{n+1-k'}^{\hat{\mathcal{I}}} , \quad (4.30)$$

where

$$a_{n,k,k'} = \delta_{k=k'} \left(h(k+1) + h(n-k+1) - \frac{1}{n+2} \right) - \frac{\delta_{k \neq k'}}{|k - k'|} + \delta_{k > k'} \frac{k+1}{(n+2)(n-k'+1)} + \delta_{k < k'} \frac{n-k+1}{(n+2)(k'+1)} , \quad (4.31)$$

$$b_{n,k,k'} = \frac{1}{n+2} \left(\delta_{k=k'} + \delta_{k > k'} \frac{n-k+1}{n-k'+1} + \delta_{k < k'} \frac{k+1}{k'+1} \right) , \quad (4.32)$$

$$c_{n,k,k'} = -((n-k+1)((k+1)b_{n+1,k+1,k'} - (k+2)b_{n+1,k,k'}) + k'b_{n,k,k'-1}) . \quad (4.33)$$

Let us now distinguish the two possible combinations of $SU(2)_L$ indices:

4.3.1 $SU(2)_L$ singlet

The Hamiltonian density can be written as

$$H'_{12} = A_{11}(0)\mathcal{P}'_{-1} + \sum_{j=1}^{\infty} \begin{pmatrix} A_{11}(j) & A_{12}(j) & A_{13}(j) \\ A_{21}(j) & A_{22}(j) & A_{23}(j) \\ A_{31}(j) & A_{32}(j) & A_{33}(j) \end{pmatrix} \mathcal{P}'_{-1-j} , \quad (4.34)$$

To fix the undetermined constants we consider the $SU(1,1)$ highest weights (of spin $-1-j$),

$$\mathcal{J}(j) = -\sum_{k=0}^j (-1)^k \binom{j}{k} \binom{j}{k} \mathcal{D}^{j-k} Q_2 \mathcal{D}^k \bar{Q}_2 , \quad (4.35)$$

$$\mathcal{K}(j) = \sqrt{2j(j+1)} \sum_{k=0}^{j-1} (-1)^k \frac{1}{k+1} \binom{j}{k} \binom{j-1}{i} \mathcal{D}^{j-k-1} \lambda_{2+} \mathcal{D}^k \bar{\lambda}_{2+} , \quad (4.36)$$

$$\bar{\mathcal{K}}(j) = -\sqrt{2j(j+1)} \sum_{k=0}^{j-1} (-1)^k \frac{1}{k+1} \binom{j}{k} \binom{j-1}{k} \mathcal{D}^{j-k-1} \bar{\lambda}_{2+} \mathcal{D}^k \lambda_{2+} . \quad (4.37)$$

These states are superconformal descendants obtained by acting with $-\frac{1}{2}\mathcal{R}_2^1\mathcal{R}_2^1$ on (B.41) for $j=0$, and with $\mathcal{Q}_+^1\tilde{\mathcal{Q}}_{+2}$ on (B.42) and (B.44) for $j>0$. The action of the Hamiltonian is, for $j>0$,

$$\begin{aligned} H'_{12}\mathcal{J}(j) &= 4\kappa^2 h(j)\mathcal{J}(j) + \frac{2\sqrt{2}}{\sqrt{j(j+1)}}\mathcal{K}(j) + \frac{2\sqrt{2}}{\sqrt{j(j+1)}}\bar{\mathcal{K}}(j) , \\ H'_{12}\mathcal{K}(j) &= 2(h(j+1) + h(j-1))\mathcal{K}(j) - \frac{2}{j(j+1)}\bar{\mathcal{K}}(j) + \frac{2\sqrt{2}}{\sqrt{j(j+1)}}\mathcal{J}(j) , \\ H'_{12}\bar{\mathcal{K}}(j) &= 2(h(j+1) + h(j-1))\bar{\mathcal{K}}(j) - \frac{2}{j(j+1)}\mathcal{K}(j) + \frac{2\sqrt{2}}{\sqrt{j(j+1)}}\mathcal{J}(j) , \end{aligned}$$

and for $j=0$,

$$H'_{12}\mathcal{J}(0) = 4\mathcal{J}(0) . \quad (4.38)$$

We can immediately read off the full one-loop Hamiltonian density in the $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$ subspace,

$$H_{12} = 4\mathcal{P}_{(-\frac{1}{2}, -\frac{1}{2})} + 2 \sum_{q=0}^{\infty} \begin{pmatrix} 2\kappa^2 h(q+1) & \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} & \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} \\ \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} & h(q+2) + h(q) & -\frac{1}{(q+1)(q+2)} \\ \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} & -\frac{1}{(q+1)(q+2)} & h(q+2) + h(q) \end{pmatrix} \mathcal{P}_{(\frac{q}{2}, \frac{q}{2})} . \quad (4.39)$$

A quick check: Let's consider the action of the Hamiltonian on the three-dimensional vector space spanned by $2\phi\bar{\phi}$, $2\bar{\phi}\phi$ and $Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{I}}\mathcal{I}}$. These are the superconformal primaries of the $q = 0$ modules. The mixing matrix is the one given in (4.39) evaluated at $q = 0$,

$$H_{12} = \begin{pmatrix} 4\kappa^2 & 2 & 2 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}, \quad (4.40)$$

again in agreement with [9].

4.3.2 $SU(2)_L$ triplet

In this case $\mathcal{H} \times \mathcal{H}$ does not mix with $\mathcal{V} \times \bar{\mathcal{V}}$ and $\bar{\mathcal{V}} \times \mathcal{V}$, and the Hamiltonian on $\mathcal{H} \times \mathcal{H}$ is simply

$$H_{12} = \sum_{q=0}^{\infty} 4\kappa^2 h(q+1) \mathcal{P}_{(\frac{q}{2}, \frac{q}{2})}. \quad (4.41)$$

5. Algebraic Evaluation of the Hamiltonian

In addition to the $SU(1,1)$ symmetry already exploited in the previous section, our closed subsector has an extra $SU(1|1) \times SU(1|1) \times U(1)$ symmetry. The generators of the two $SU(1|1)$ s are

$$B = \frac{1}{2}\mathcal{L}_-^- + \frac{1}{2}\dot{\mathcal{L}}_-^{\dot{-}} + \frac{1}{2}D_0 + r, \quad \mathcal{S}(g) = S_1^-(g), \quad \mathcal{Q}(g) = Q_-^{-1}(g), \quad (5.1)$$

$$\tilde{B} = \frac{1}{2}\mathcal{L}_-^- + \frac{1}{2}\dot{\mathcal{L}}_-^{\dot{-}} + \frac{1}{2}D_0 - r, \quad \tilde{\mathcal{S}}(g) = \tilde{S}^{\dot{-}2}(g), \quad \tilde{\mathcal{Q}}(g) = \tilde{Q}_{-2}(g), \quad (5.2)$$

and can be checked to commute with the $SU(1,1)$ generators (3.13). The $U(1)$ is a central element corresponding to the quantum part of the dilatation operator, $\delta D(g)$.

The (anti)commutators are

$$[B, \mathcal{Q}(g)] = \mathcal{Q}(g), \quad [\tilde{B}, \tilde{\mathcal{Q}}(g)] = \tilde{\mathcal{Q}}(g), \quad (5.3)$$

$$[B, \mathcal{S}(g)] = -\mathcal{S}(g), \quad [\tilde{B}, \tilde{\mathcal{S}}(g)] = -\tilde{\mathcal{S}}(g), \quad (5.4)$$

$$\{\mathcal{S}(g), \mathcal{Q}(g)\} = \frac{1}{2}\delta D(g), \quad \{\tilde{\mathcal{S}}(g), \tilde{\mathcal{Q}}(g)\} = \frac{1}{2}\delta D(g). \quad (5.5)$$

The operator $L = B + \tilde{B}$ evaluates to 1 on each of the elementary letters of the subsector, and thus measures the “length” of a state. Since

$$[L, \mathcal{Q}(g)] = \mathcal{Q}(g), \quad [L, \tilde{\mathcal{Q}}(g)] = \tilde{\mathcal{Q}}(g), \quad (5.6)$$

$$[L, \mathcal{S}(g)] = -\mathcal{S}(g), \quad [L, \tilde{\mathcal{S}}(g)] = -\tilde{\mathcal{S}}(g). \quad (5.7)$$

we learn that $\mathcal{Q}(g)$ and $\tilde{\mathcal{Q}}(g)$ increase the length of a state by one unit while $\mathcal{S}(g)$ and $\tilde{\mathcal{S}}(g)$ decrease it.

5.1 First order expressions for $\mathcal{Q}(g)$ and $\mathcal{S}(g)$

In the classical limit $g \rightarrow 0$ one easily checks that the $SU(1|1)$ generators annihilate all the states of the subsector, consistent with the fact that they must change the length of a state. As in [13], we know that there must be quantum corrections to $\mathcal{Q}(g)$ and $\mathcal{S}(g)$, because their anticommutator must yield a non-vanishing quantum dilation operator. Writing $\mathcal{Q}(g) = g\mathcal{Q} + O(g^2)$, the most general ansatz for the action of \mathcal{Q} on λ compatible with Lorentz and R-charge conservation is

$$\begin{aligned} \mathcal{Q}\lambda_n &= \sum_{k'=0}^n a_{n,k'} \mathcal{Q}_{k'} \bar{\mathcal{Q}}_{n-k'} \\ &+ \sum_{k'=0}^{n-1} b_{n,k'} \lambda_{k'} \bar{\lambda}_{n-k'-1} + \sum_{k'=0}^{n-1} c_{n,k'} \bar{\lambda}_{k'} \lambda_{n-k'-1} \end{aligned} \quad (5.8)$$

for arbitrary coefficients $a_{n,k'}$, $b_{n,k'}$ and $c_{n,k'}$. The coefficients can be constrained by requiring that \mathcal{Q} commutes with the $SU(1,1)$ algebra. Requiring $[\mathcal{J}', \mathcal{Q}]\lambda_n = 0$ fixes $a_{n,k'}$ to be a constant $a_{n,k'} = \alpha'$, and $b_{n,k'} = c_{n,k'} = 0$. This is however too restrictive, and as in $\mathcal{N} = 4$ SYM [13], one should only require that $[\mathcal{J}', \mathcal{Q}]$ annihilates all gauge invariant states (closed spin chains). We should demand $[\mathcal{J}', \mathcal{Q}]\lambda_n \sim 0$, where \sim stands for equivalence up to a gauge transformation. There are two independent gauge transformations, corresponding to adding an extra $\bar{\lambda}$ or $\check{\bar{\lambda}}$ to the chain, so we impose

$$[\mathcal{Q}, \mathcal{J}'_+]\lambda_n = \alpha (\lambda_n \bar{\lambda} + \bar{\lambda} \lambda_n) , \quad (5.9)$$

$$[\mathcal{Q}, \mathcal{J}'_+]\bar{\lambda}_n = \alpha (\bar{\lambda}_n \bar{\lambda} + \bar{\lambda} \bar{\lambda}_n) , \quad (5.10)$$

$$[\mathcal{Q}, \mathcal{J}'_+]\mathcal{Q}_n = \alpha (\bar{\lambda} \mathcal{Q}_n - \gamma \mathcal{Q}_n \bar{\lambda}) , \quad (5.11)$$

$$[\mathcal{Q}, \mathcal{J}'_+]\bar{\mathcal{Q}}_n = \alpha (\gamma \bar{\lambda} \bar{\mathcal{Q}}_n - \bar{\mathcal{Q}}_n \bar{\lambda}) , \quad (5.12)$$

$$[\mathcal{Q}, \mathcal{J}'_+]\check{\lambda}_n = \alpha \gamma (\check{\lambda}_n \bar{\lambda} + \bar{\lambda} \check{\lambda}_n) , \quad (5.13)$$

$$[\mathcal{Q}, \mathcal{J}'_+]\check{\bar{\lambda}}_n = \alpha \gamma (\check{\bar{\lambda}}_n \bar{\lambda} + \bar{\lambda} \check{\bar{\lambda}}_n) , \quad (5.14)$$

where we have labelled by α and $\alpha\gamma$ the two independent gauge parameters. We now find

$$a_{n,k'} = \alpha' , \quad (5.15)$$

$$b_{n,k'} = \frac{\alpha}{n - k'} , \quad (5.16)$$

$$c_{n,k'} = \frac{\alpha}{k' + 1} , \quad (5.17)$$

where at this stage α and α' are arbitrary constants. Similarly, for the action on the other

states of the sector,

$$\begin{aligned} \mathcal{Q}\check{\lambda}_n &= \sum_{k'=0}^n \alpha'' \bar{Q}_{k'} Q_{n-k'} \\ &+ \alpha\gamma \left(\sum_{k'=0}^{n-1} \frac{1}{n-k'} \check{\lambda}_{k'} \bar{\lambda}_{n-k'-1} + \sum_{k'=0}^{n-1} \frac{1}{k'+1} \bar{\lambda}_{k'} \check{\lambda}_{n-k'-1} \right), \end{aligned} \quad (5.18)$$

$$\mathcal{Q}\bar{\lambda}_n = \alpha \sum_{k'=0}^{n-1} \frac{n+1}{(k'+1)(n-k')} \bar{\lambda}_{k'} \bar{\lambda}_{n-k'-1}, \quad (5.19)$$

$$\mathcal{Q}\bar{\bar{\lambda}}_n = \alpha\gamma \sum_{k'=0}^{n-1} \frac{n+1}{(k'+1)(n-k')} \bar{\bar{\lambda}}_{k'} \bar{\bar{\lambda}}_{n-k'-1}, \quad (5.20)$$

$$\mathcal{Q}Q_n = \alpha \sum_{k'=0}^{n-1} \left(\frac{1}{k'+1} \bar{\lambda}_{k'} Q_{n-k'-1} - \frac{\gamma}{n-k'} Q_{k'} \bar{\lambda}_{n-k'-1} \right), \quad (5.21)$$

$$\mathcal{Q}\bar{Q}_n = \alpha \sum_{k'=0}^{n-1} \left(\frac{\gamma}{k'+1} \bar{\bar{\lambda}}_{k'} \bar{Q}_{n-k'-1} - \frac{1}{n-k'} \bar{Q}_{k'} \bar{\lambda}_{n-k'-1} \right). \quad (5.22)$$

One can check that the commutators $[\mathcal{J}'_-, \mathcal{Q}] = 0$ and $[\mathcal{J}'_3, \mathcal{Q}] = 0$ are then identically satisfied with the action of \mathcal{Q} given by the above expressions. An analogous analysis can be performed for \mathcal{S} . Now the relevant gauge transformations are

$$[\mathcal{S}, \mathcal{J}'_-] \bar{\lambda}_k \bar{\lambda}_{n-k} = \beta (\delta_{k=0} + \delta_{n=k}) \bar{\lambda}_n, \quad [\mathcal{S}, \mathcal{J}'_-] \bar{\bar{\lambda}}_k \bar{\bar{\lambda}}_{n-k} = \beta\gamma' (\delta_{k=0} + \delta_{n=k}) \bar{\bar{\lambda}}_n, \quad (5.23)$$

$$[\mathcal{S}, \mathcal{J}'_-] \lambda_k \bar{\lambda}_{n-k} = \beta \delta_{n=k} \lambda_n, \quad [\mathcal{S}, \mathcal{J}'_-] \check{\lambda}_k \bar{\bar{\lambda}}_{n-k} = \beta\gamma' \delta_{n=k} \check{\lambda}_n, \quad (5.24)$$

$$[\mathcal{S}, \mathcal{J}'_-] \bar{\lambda}_k \lambda_{n-k} = \beta \delta_{k=0} \lambda_n, \quad [\mathcal{S}, \mathcal{J}'_-] \bar{\bar{\lambda}}_k \check{\lambda}_{n-k} = \beta\gamma' \delta_{k=0} \check{\lambda}_n, \quad (5.25)$$

$$[\mathcal{S}, \mathcal{J}'_-] \bar{\lambda}_k Q_{n-k} = \beta \delta_{k=0} Q_n, \quad [\mathcal{S}, \mathcal{J}'_-] \bar{\bar{\lambda}}_k \bar{Q}_{n-k} = \beta\gamma' \delta_{k=0} \bar{Q}_n, \quad (5.26)$$

$$[\mathcal{S}, \mathcal{J}'_-] \bar{Q}_k \bar{\lambda}_{n-k} = -\beta \delta_{n=k} \bar{Q}_n, \quad [\mathcal{S}, \mathcal{J}'_-] Q_k \bar{\bar{\lambda}}_{n-k} = -\beta\gamma' \delta_{n=k} Q_n, \quad (5.27)$$

and the action of \mathcal{S} consistent with them is

$$\mathcal{S} Q_{k\hat{I}} \bar{Q}_{n-k}^{\hat{J}} = \frac{\beta'}{n+1} \lambda_n \delta_{\hat{I}}^{\hat{J}}, \quad \mathcal{S} \bar{Q}_k^{\hat{J}} Q_{n-k\hat{I}} = \frac{\beta''}{n+1} \check{\lambda}_n \delta_{\hat{I}}^{\hat{J}}, \quad (5.28)$$

$$\mathcal{S} \bar{\lambda}_k \bar{\lambda}_{n-k} = \beta \bar{\lambda}_{n+1}, \quad \mathcal{S} \bar{\bar{\lambda}}_k \bar{\bar{\lambda}}_{n-k} = \gamma' \beta \bar{\bar{\lambda}}_{n+1}, \quad (5.29)$$

$$\mathcal{S} \lambda_k \bar{\lambda}_{n-k} = \beta \frac{k+1}{n+2} \lambda_{n+1}, \quad \mathcal{S} \check{\lambda}_k \bar{\bar{\lambda}}_{n-k} = \gamma' \beta \frac{k+1}{n+2} \check{\lambda}_{n+1}, \quad (5.30)$$

$$\mathcal{S} \bar{\lambda}_k \lambda_{n-k} = \beta \frac{n-k+1}{n+2} \lambda_{n+1}, \quad \mathcal{S} \bar{\bar{\lambda}}_k \check{\lambda}_{n-k} = \gamma' \beta \frac{n-k+1}{n+2} \check{\lambda}_{n+1}, \quad (5.31)$$

$$\mathcal{S} \bar{\lambda}_k Q_{n-k} = \beta Q_{n+1}, \quad \mathcal{S} Q_k \bar{\bar{\lambda}}_{n-k} = -\gamma' \beta Q_{n+1}, \quad (5.32)$$

$$\mathcal{S} \bar{Q}_k \bar{\lambda}_{n-k} = -\beta \bar{Q}_{n+1}, \quad \mathcal{S} Q_k \bar{\bar{\lambda}}_{n-k} = -\gamma' \beta Q_{n+1}. \quad (5.33)$$

With these expressions, the remaining commutators $[\mathcal{J}'_+, \mathcal{S}] = 0$ and $[\mathcal{J}'_3, \mathcal{S}] = 0$ are automatically satisfied.

As we are interested in unitary representations of the superconformal algebra, we impose the hermiticity condition⁷

$$\mathcal{Q}^\dagger = \mathcal{S}, \quad (5.34)$$

which implies the following reality constraints for the undetermined coefficients:

$$\alpha = \beta^*, \quad (5.35)$$

$$\alpha' = \beta'^*, \quad (5.36)$$

$$\alpha'' = \beta''^*, \quad (5.37)$$

$$\gamma = \gamma'^*. \quad (5.38)$$

Having determined the $O(g)$ action of $\mathcal{Q}(g)$ and $\mathcal{S}(g)$, we are now in the position to evaluate the one-loop Hamiltonian, since the algebra (5.5) implies

$$H' = 2\{\mathcal{S}, \mathcal{Q}\}. \quad (5.39)$$

Let us proceed to find H' in the different subspaces:

5.2 $\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$

The $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$ case is identical with $\mathcal{N} = 4$, we refer the interested reader to [13] for details of the calculation. The result is

$$H'_{12} \bar{\lambda}_k \bar{\lambda}_{n-k} = 2|\alpha|^2 \sum_{k'=0}^n c_{n,k,k'} \bar{\lambda}_{k'} \bar{\lambda}_{n-k'}, \quad (5.40)$$

with

$$c_{n,k,k'} = \delta_{k=k'} (h(k+1) + h(n-k+1)) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k>k'}}{n-k'+1} + \frac{\delta_{k<k'}}{k'+1}. \quad (5.41)$$

For $\mathcal{V} \times \mathcal{V}$ the calculation is very similar, and the result is

$$H'_{12} \lambda_k \lambda_{n-k} = 2|\alpha|^2 \sum_{k'=0}^n c_{n,k,k'} \lambda_{k'} \lambda_{n-k'}, \quad (5.42)$$

with

$$c_{n,k,k'} = \delta_{k=k'} \left(h(k+1) + h(n-k+1) + \frac{|\alpha'|^2}{|\alpha|^2} - 1 \right) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k>k'}}{n-k'+1} + \frac{\delta_{k<k'}}{k'+1}. \quad (5.43)$$

⁷To exhibit hermiticity explicitly one needs to rescale the fermion letters as $\chi_n \rightarrow \frac{\chi_n}{\sqrt{n+1}}$, where χ_n stands for $\lambda_n, \check{\lambda}_n, \bar{\lambda}_n$ or $\bar{\check{\lambda}}_n$.

We now impose the physical requirement that the action of the Hamiltonian on $\lambda\lambda$ is identical to the action on $\bar{\lambda}\bar{\lambda}$ (this is CPT invariance in the field theory). This fixes $|\alpha'|^2 = |\alpha|^2$, which implies $\alpha' = e^{i\theta_1}\alpha$, where θ_1 is an arbitrary phase. Proceeding just as in Section 4.1 we can uplift the result to the full theory,

$$H_{12} = 0 \times \mathcal{P}_{\bar{\varepsilon}} + |\alpha|^2 \sum_{q=0}^{\infty} 4h(q+1) \mathcal{P}_{(\frac{q+1}{2}, \frac{q-1}{2})}. \quad (5.44)$$

The overall constant $|\alpha|^2$ cannot be fixed algebraically and is related to a rescaling of the coupling. To match with the field theory result (4.6) we need to set $|\alpha|^2 = 1$.

5.3 $\bar{\mathcal{V}} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \bar{\mathcal{V}}$

Since this case is somewhat different from $\mathcal{N} = 4$ SYM because of multiplet mixing, let us give a few more details of the calculation. We need to evaluate

$$H'_{12} \lambda_k Q_{n-k} = 2(\mathcal{S}\mathcal{Q} + \mathcal{Q}\mathcal{S}) \lambda_k Q_{n-k}. \quad (5.45)$$

In the first term inside the parenthesis we can act with \mathcal{Q} in either the first or the second site, we will denote this contributions by \mathcal{Q}_1 and \mathcal{Q}_2 . Both choices will increase the length of the chain by one, which implies that \mathcal{S} can act in either sites 1-2 or 2-3, we will denote this by \mathcal{S}_{12} and \mathcal{S}_{23} . Taking into account all possible combinations the action of the Hamiltonian is

$$H'_{12} = 2(\mathcal{S}_{12}\mathcal{Q}_1 + \mathcal{S}_{23}\mathcal{Q}_1 + \mathcal{S}_{12}\mathcal{Q}_2 + \mathcal{S}_{23}\mathcal{Q}_2 + \mathcal{Q}_1\mathcal{S}_{12}). \quad (5.46)$$

Each individual contribution can be calculated by straightforward application of the action of the supercharges given in the previous section,

$$\mathcal{S}_{12}\mathcal{Q}_1 \bar{\lambda}_k Q_{n-k} = 4h(k) \bar{\lambda}_k Q_{n-k}, \quad (5.47)$$

$$\mathcal{S}_{23}\mathcal{Q}_1 \bar{\lambda}_k Q_{n-k} = -2 \sum_{k'=0}^{k-1} \left(\frac{1}{k'+1} + \frac{1}{k-k'} \right) \bar{\lambda}_{k'} Q_{n-k'}, \quad (5.48)$$

$$\mathcal{S}_{12}\mathcal{Q}_2 \bar{\lambda}_k Q_{n-k} = -2 \sum_{k'=k+1}^n \left(\frac{1}{k'-k} \bar{\lambda}_{k'} Q_{n-k'} + \frac{\gamma}{n-k'+1} Q_{k'} \bar{\lambda}_{n-k'} \right), \quad (5.49)$$

$$\mathcal{S}_{23}\mathcal{Q}_2 \bar{\lambda}_k Q_{n-k} = 2(1 + |\gamma|^2) h(n-k) \bar{\lambda}_k Q_{n-k}, \quad (5.50)$$

$$\mathcal{Q}_1\mathcal{S}_{12} \bar{\lambda}_k Q_{n-k} = 2 \sum_{k'=0}^n \left(\frac{1}{k'+1} \bar{\lambda}_{k'} Q_{n-k'} - \frac{\gamma}{n-k'+1} Q_{k'} \bar{\lambda}_{n-k'} \right). \quad (5.51)$$

Now, since $\mathcal{S}_{12}\mathcal{Q}_1$ and $\mathcal{S}_{23}\mathcal{Q}_2$ act at the single site level, they are analogous to the self-energy contributions in the field theory calculation. As usual for spin chains, we distribute them evenly in two adjacent sites by adding an extra factor of one half. An analogous calculation

can be done for $2\{\mathcal{S}, \mathcal{Q}\}Q_k\check{\lambda}_{n-k}$, the action of the Hamiltonian in this subspace is

$$H'_{12}\lambda_k Q_{n-k} = 2 \sum_{k'=0}^n a_{n,k,k'} \lambda_{k'} Q_{n-k'} + 2 \sum_{k'=0}^n b_{n,k,k'} Q_{k'} \check{\lambda}_{n-k'}, \quad (5.52)$$

$$H'_{12}Q_k \check{\lambda}_{n-k} = 2 \sum_{k'=0}^n \check{a}_{n,k,k'} Q_{k'} \check{\lambda}_{n-k'} + 2 \sum_{k'=0}^n \check{b}_{n,k,k'} \lambda_{k'} Q_{n-k'}, \quad (5.53)$$

where

$$a_{n,k,k'} = \delta_{k=k'} \left(h(k+1) + \frac{1+|\gamma|^2}{2} h(n-k) \right) - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k < k'}}{k'+1}, \quad (5.54)$$

$$b_{n,k,k'} = -\gamma \frac{\delta_{k \geq k'}}{n-k'+1}, \quad (5.55)$$

$$\check{a}_{n,k,k'} = \frac{1+|\gamma|^2}{2} h(k) \delta_{k=k'} + |\gamma|^2 \left(h(n-k+1) \delta_{k=k'} - \frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k > k'}}{n-k'+1} \right), \quad (5.56)$$

$$\check{b}_{n,k,k'} = -\gamma^* \frac{\delta_{k \leq k'}}{k'+1}. \quad (5.57)$$

This expression for the two-site Hamiltonian has the same problem we encountered in the Feynman diagram calculation: it is not $SU(1,1)$ invariant. But it *can* be made invariant by performing the gauge transformation (4.14), now with

$$f(k) = (1 - |\gamma|^2) h(k). \quad (5.58)$$

The uplifting to the full theory works exactly as in Section 4.2. Defining $\gamma \equiv \eta e^{i\theta_2}$, where η and θ_2 are real parameters, we find

$$H_{12} = 2 \sum_{q=-1}^{\infty} \begin{pmatrix} h(q+2) + h(q+1) & -\frac{\eta}{q+2} e^{i\theta_2} \\ -\frac{\eta}{q+2} e^{-i\theta_2} & \eta^2 (h(q+2) + h(q+1)) \end{pmatrix} \mathcal{P}_{(\frac{q+1}{2}, \frac{q}{2})}. \quad (5.59)$$

The phase θ_2 does not enter in any physical anomalous dimension, and can in fact be set to zero by a similarity transformation. Comparison with (4.23) shows then perfect agreement with the field theory calculation, if we identify $\eta \equiv \kappa$.

5.4 $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$

Following similar steps as in the previous subsection, we obtain for this subspace

$$H'_{12} Q_k \hat{\mathcal{I}} \bar{Q}_{n-k}^{\hat{\mathcal{J}}} = 2 \sum_{k'=0}^n (a_{n,k,k'})_{\hat{\mathcal{I}} \hat{\mathcal{L}}}^{\hat{\mathcal{J}} \hat{\mathcal{K}}} Q_{k'} \hat{\mathcal{K}} \bar{Q}_{n-k'}^{\hat{\mathcal{L}}} + \delta_{\hat{\mathcal{I}}}^{\hat{\mathcal{J}}} 2 \sum_{k'=0}^{n-1} (b_{n,k,k'} \lambda_{k'} \bar{\lambda}_{n-k'-1} + c_{n,k,k'} \bar{\lambda}_{k'} \lambda_{n-k'-1}) , \quad (5.60)$$

with

$$a_{n,k,k'} = \frac{\hat{\mathbb{K}}}{(n+1)} + \kappa^2 \hat{\mathbb{I}} \left(\delta_{k=k'} (h(k) + h(n-k)) - \frac{\delta_{k \neq k'}}{|k-k'|} \right), \quad (5.61)$$

$$b_{n,k,k'} = \frac{e^{-i\theta_1}}{n+1} \left(-\frac{\delta_{k>k'}}{n-k'} + \frac{\delta_{k \leq k'}}{k'+1} \right), \quad (5.62)$$

$$c_{n,k,k'} = -\frac{e^{-i\theta_1}}{(n+1)} \left(-\frac{\delta_{k>k'}}{n-k'} + \frac{\delta_{k \leq k'}}{k'+1} \right). \quad (5.63)$$

This expressions precisely coincide with our previous Feynman diagram results (4.26)-(4.28) apart from the extra $e^{-i\theta_1}$ phases in the cross terms. For the action on the fermions, we get

$$H_{12} \lambda_k \bar{\lambda}_{n-k} = 2 \sum_{k'=0}^n (a_{n,k,k'} \lambda_{k'} \bar{\lambda}_{n-k'} + b_{n,k,k'} \bar{\lambda}_{k'} \lambda_{n-k'}) + 2 \sum_{k'=0}^{n+1} c_{n,k,k'} Q_{k'} \hat{\mathbb{I}} \bar{Q}_{n+1-k'}, \quad (5.64)$$

$$H_{12} \bar{\lambda}_k \lambda_{n-k} = 2 \sum_{k'=0}^n (a_{n,k,k'} \bar{\lambda}_{k'} \lambda_{n-k'} + b_{n,k,k'} \lambda_{k'} \bar{\lambda}_{n-k'}) - 2 \sum_{k'=0}^{n+1} c_{n,k,k'} Q_{k'} \hat{\mathbb{I}} \bar{Q}_{n+1-k'}, \quad (5.65)$$

where

$$a_{n,k,k'} = \delta_{k=k'} \left(h(k+1) + h(n-k+1) - \frac{1}{n+2} \right) - \frac{\delta_{k \neq k'}}{|k-k'|} + \delta_{k>k'} \frac{k+1}{(n+2)(n-k'+1)} + \delta_{k<k'} \frac{n-k+1}{(n+2)(k'+1)}, \quad (5.66)$$

$$b_{n,k,k'} = \frac{e^{i\theta_1}}{n+2} \left(\delta_{k=k'} + \delta_{k>k'} \frac{n-k+1}{n-k'+1} + \delta_{k<k'} \frac{k+1}{k'+1} \right), \quad (5.67)$$

$$c_{n,k,k'} = -e^{i\theta_1} \left(-\delta_{k \geq k'} + \frac{k+1}{n+2} \right), \quad (5.68)$$

again in agreement with the Feynman diagram result (4.31)-(4.33) up to the extra $e^{i\theta_1}$ factors. (Note that $c_{n,k,k'}$ coefficients in (4.33) and in (5.68) are equal, thanks to a non-trivial identity.)

Uplifting to the full theory gives

$$H_{12} = 4\mathcal{P}_{(-\frac{1}{2}, -\frac{1}{2})} + 2 \sum_{q=0}^{\infty} \begin{pmatrix} 2\kappa^2 h(q+1) & \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} e^{-i\theta_1} & \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} e^{-i\theta_1} \\ \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} e^{i\theta_1} & h(q+2) + h(q) & -\frac{1}{(q+1)(q+2)} \\ \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}} e^{i\theta_1} & -\frac{1}{(q+1)(q+2)} & h(q+2) + h(q) \end{pmatrix} \mathcal{P}_{(\frac{q}{2}, \frac{q}{2})}. \quad (5.69)$$

The phase θ_1 can be set to zero by a similarity transformation, and we find perfect agreement with the field theory answer (4.39).

6. The Harmonic Action

While we have obtained an explicit expression for the full one-loop Hamiltonian in terms of superconformal projectors, evaluating this expression on concrete states is still a rather cumbersome procedure. For $\mathcal{N} = 4$ SYM Beisert [12] was able to find a very explicit and elegant formula for the action of the Hamiltonian on any state, using the oscillator representation, which he called the “harmonic action”. Beisert’s approach easily generalizes to our case and allows to write a harmonic action for the interpolating SCFT.

6.1 $\mathcal{V} \times \mathcal{V}$

For a state in $\mathcal{V} \times \mathcal{V}$ we found that the action of the Hamiltonian is identical with that of $\mathcal{N} = 4$ SYM. Let’s review then how the harmonic action works in this case. As pointed out in [12] a general state in $\mathcal{V} \times \mathcal{V}$ can be written as

$$|s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \mathcal{V}} = A_{s_1 A_1}^\dagger \dots A_{s_n A_n}^\dagger |0\rangle \otimes |0\rangle, \quad (6.1)$$

where $A_A^\dagger = (\mathbf{a}_A^\dagger, \mathbf{b}_A^\dagger, \mathbf{c}_A^\dagger)$ and $s_i = 1, 2$ indicates in which site the oscillator sits. The action of the Hamiltonian on this state does not change the number of oscillators but merely shifts them from site 1 to site 2 (or vice versa) in all possible combinations. This can be written as

$$H_{12}|s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \mathcal{V}} = \sum_{s'_1, \dots, s'_n} c_{n, n_{12}, n_{21}} \delta_{C_1, 0} \delta_{C_2, 0} |s'_1, \dots, s'_n; A\rangle_{\mathcal{V} \times \mathcal{V}}, \quad (6.2)$$

where the delta functions project onto states with zero central charge and n_{ij} counts the number of oscillators moving from site i to site j . The explicit formula for the function $c_{n, n_{12}, n_{21}}$ is

$$c_{n, n_{12}, n_{21}} = (-1)^{1+n_{12}n_{21}} \frac{\Gamma(\frac{1}{2}(n_{12} + n_{21}))\Gamma(1 + \frac{1}{2}(n - n_{12} - n_{21}))}{\Gamma(1 + \frac{1}{2}n)}, \quad (6.3)$$

with $c_{n, 0, 0} = h(\frac{n}{2})$. In [12] it was proven that this function is a superconformal invariant and that it has the appropriate eigenvalues when acting on the $\hat{\mathcal{C}}_{0(\frac{q+1}{2}, \frac{q-1}{2})}$ modules, namely

$$H_{12}\hat{\mathcal{C}}_{0(\frac{q+1}{2}, \frac{q-1}{2})} = 2h(q+1)\hat{\mathcal{C}}_{0(\frac{q+1}{2}, \frac{q-1}{2})}. \quad (6.4)$$

6.2 $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$

General states in $\mathcal{V} \times \mathcal{H}$ and $\mathcal{H} \times \mathcal{V}$ can be written as

$$|s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \mathcal{H}} = A_{s_1 A_1}^\dagger \dots A_{s_n A_n}^\dagger |0\rangle \otimes |\mathbf{d}\rangle, \quad (6.5)$$

$$|s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{V}} = A_{s_1 A_1}^\dagger \dots A_{s_n A_n}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle, \quad (6.6)$$

where $|\mathbf{d}\rangle = \mathbf{d}^\dagger|0\rangle$. We claim that the action of H_{12} is given by ⁸

$$\begin{aligned} H_{12}|s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \mathcal{H}} &= \sum_{s'_1, \dots, s'_n} c_{n+1, n_{12}, n_{21}} |s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \mathcal{H}} \\ &+ \kappa \sum_{s'_1, \dots, s'_n} c_{n+1, n_{12}, n_{21}+1} |s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \tilde{\mathcal{V}}} \end{aligned} \quad (6.7)$$

and

$$\begin{aligned} H_{12}|s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \tilde{\mathcal{V}}} &= \kappa^2 \sum_{s'_1, \dots, s'_n} c_{n+1, n_{12}, n_{21}} |s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \tilde{\mathcal{V}}} \\ &- \kappa \sum_{s'_1, \dots, s'_n} c_{n+1, n_{12}, n_{21}+1} |s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \mathcal{H}} \end{aligned} \quad (6.8)$$

Invariance under the superconformal group is guaranteed by the same arguments given in [12]. The only thing we need to check is that this expression correctly reproduces the 2×2 matrix given in (4.23). This can be easily done with an algebra software like Mathematica.

Let us work out an example. Consider the action of the Hamiltonian on $\lambda_{\mathcal{I}} Q_{\mathcal{J}}$ (Lorentz and $SU(2)_L$ indices are open and go along for the ride). First, we need to write the state in a “canonical order” to make sure all our signs are correct,

$$\lambda_{\mathcal{I}} Q_{\mathcal{J}} = \mathbf{a}_{(1)}^\dagger \mathbf{c}_{(1)\mathcal{I}}^\dagger |0\rangle \otimes \mathbf{c}_{(2)\mathcal{J}}^\dagger |\mathbf{d}\rangle = \mathbf{a}_{(1)}^\dagger \mathbf{c}_{(1)\mathcal{I}}^\dagger \mathbf{c}_{(2)\mathcal{J}}^\dagger |0\rangle \otimes |\mathbf{d}\rangle, \quad (6.9)$$

$$Q_{\mathcal{I}} \tilde{\lambda}_{\mathcal{J}} = \mathbf{c}_{(1)\mathcal{I}}^\dagger |\mathbf{d}\rangle \otimes \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(2)\mathcal{J}}^\dagger |\check{0}\rangle = -\mathbf{c}_{(1)\mathcal{I}}^\dagger \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(2)\mathcal{J}}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle. \quad (6.10)$$

For $\lambda_1 Q_1$, the action of the Hamiltonian is

$$\begin{aligned} H_{12} \lambda_1 Q_1 &= c_{4,0,0} \mathbf{a}_{(1)}^\dagger \mathbf{c}_{(1)1}^\dagger \mathbf{c}_{(2)1}^\dagger |0\rangle \otimes |\mathbf{d}\rangle + c_{4,1,1} \mathbf{a}_{(1)}^\dagger \mathbf{c}_{(2)1}^\dagger \mathbf{c}_{(1)1}^\dagger |0\rangle \otimes |\mathbf{d}\rangle \\ &+ \kappa \left(c_{4,1,1} \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(1)1}^\dagger \mathbf{c}_{(2)1}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle + c_{4,2,2} \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(2)1}^\dagger \mathbf{c}_{(1)1}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle \right) \\ &= \lambda_1 Q_1 - \kappa Q_1 \tilde{\lambda}_1, \end{aligned} \quad (6.11)$$

while for $\lambda_1 Q_2$,

$$\begin{aligned} H_{12} \lambda_1 Q_2 &= c_{4,0,0} \mathbf{a}_{(1)}^\dagger \mathbf{c}_{(1)1}^\dagger \mathbf{c}_{(2)2}^\dagger |0\rangle \otimes |\mathbf{d}\rangle + c_{4,1,1} \mathbf{a}_{(1)}^\dagger \mathbf{c}_{(2)1}^\dagger \mathbf{c}_{(1)1}^\dagger |0\rangle \otimes |\mathbf{d}\rangle + c_{4,1,1} \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(1)1}^\dagger \mathbf{c}_{(1)1}^\dagger |0\rangle \otimes |\mathbf{d}\rangle \\ &+ \kappa c_{4,1,1} \left(\mathbf{a}_{(1)}^\dagger \mathbf{c}_{(2)1}^\dagger \mathbf{c}_{(2)2}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle + \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(1)1}^\dagger \mathbf{c}_{(2)2}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle \right) \\ &+ \kappa c_{4,2,2} \mathbf{a}_{(2)}^\dagger \mathbf{c}_{(2)1}^\dagger \mathbf{c}_{(1)2}^\dagger |\mathbf{d}\rangle \otimes |\check{0}\rangle \\ &= \frac{3}{2} \lambda_1 Q_2 - \frac{1}{2} \lambda_2 Q_1 + \frac{1}{2} \phi \psi - \frac{\kappa}{2} (Q_1 \tilde{\lambda}_2 + Q_2 \tilde{\lambda}_1 - \psi \check{\phi}). \end{aligned} \quad (6.12)$$

⁸To simplify the notation we will omit the delta functions $\delta_{C_1,0} \delta_{C_2,0}$.

Similar calculations can be done for $\lambda_2 Q_1$ and $\lambda_2 Q_2$. The final result is

$$H_{12}\lambda_{\mathcal{I}}Q_{\mathcal{J}} = \frac{3}{2}\lambda_{\mathcal{I}}Q_{\mathcal{J}} - \frac{1}{2}\lambda_{\mathcal{J}}Q_{\mathcal{I}} - \frac{1}{2}\epsilon_{\mathcal{I}\mathcal{J}}\phi\tilde{\psi} - \frac{\kappa}{2}Q_{\mathcal{I}}\tilde{\lambda}_{\mathcal{J}} - \frac{\kappa}{2}Q_{\mathcal{J}}\tilde{\lambda}_{\mathcal{I}} - \frac{\kappa}{2}\epsilon_{\mathcal{I}\mathcal{J}}\tilde{\psi}\check{\phi}, \quad (6.13)$$

which is consistent with the explicit Feynman diagram calculations of Appendix D.

6.3 $\mathcal{H} \times \mathcal{H} \leftrightarrow \mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V}$

For these multiplets we have the following states

$$|s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}} = A_{s_1 A_1}^\dagger \dots A_{s_n A_n}^\dagger |\mathbf{d}\rangle \otimes |\tilde{\mathbf{d}}\rangle, \quad (6.14)$$

$$|s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} = A_{s_1 A_1}^\dagger \dots A_{s_n A_n}^\dagger |\mathbf{d}\tilde{\mathbf{d}}\rangle \otimes |0\rangle, \quad (6.15)$$

$$|s_1, \dots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}} = A_{s_1 A_1}^\dagger \dots A_{s_n A_n}^\dagger |0\rangle \otimes |\mathbf{d}\tilde{\mathbf{d}}\rangle. \quad (6.16)$$

Let us consider the $SU(2)_L$ triplet and singlet cases separately. We have found:

6.3.1 $SU(2)_L$ singlet

$$\begin{aligned} H_{12}|s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}} &= \sum_{s'_1, \dots, s'_n} (\kappa^2 c_{n, n_{12}, n_{21}} - 2c_{n+2, n_{12}+2, n_{21}}) |s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}} \\ &+ 2 \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}, n_{21}+1} |s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} \\ &+ 2 \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}+1, n_{21}} |s_1, \dots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}}, \end{aligned} \quad (6.17)$$

$$\begin{aligned} H_{12}|s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} &= \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}, n_{21}} |s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} \\ &+ \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}+2, n_{21}} |s_1, \dots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}} \\ &+ \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}+1, n_{21}} |s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}}, \end{aligned} \quad (6.18)$$

$$\begin{aligned} H_{12}|s_1, \dots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}} &= \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}, n_{21}} |s_1, \dots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}} \\ &+ \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}, n_{21}+2} |s_1, \dots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} \\ &+ \sum_{s'_1, \dots, s'_n} c_{n+2, n_{12}, n_{21}+1} |s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}}. \end{aligned} \quad (6.19)$$

6.3.2 $SU(2)_L$ triplet

$$H_{12}|s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}} = \kappa^2 \sum_{s'_1, \dots, s'_n} c_{n, n_{12}, n_{21}} |s_1, \dots, s_n; A\rangle_{\mathcal{H} \times \mathcal{H}}. \quad (6.20)$$

7. Discussion

$\mathcal{N} = 2$ superconformal symmetry turns out to be more constraining than naively expected: it fixes the one-loop Hamiltonian of $\mathcal{N} = 2$ SCQCD completely, and that of the interpolating quiver theory up to a single parameter. Knowledge of the full Hamiltonian should allow to settle the question of one-loop integrability for the $\mathcal{N} = 2$ SCQCD spin chain. The question is really whether the *full* spin chain is integrable. One-loop integrable subsectors are easy to identify, but those are trivially isomorphic to analogous sectors of $\mathcal{N} = 4$ SYM. Two notable examples of one-loop integrable subsectors are the $SU(2|1)$ sector spanned by the letters $\{\phi, \lambda_{1\alpha}\}$, and the $SU(2, 1|2)$ sector spanned by the letters $\{\mathcal{D}_{+\dot{\alpha}}^k \phi, \mathcal{D}_{+\dot{\alpha}}^k \lambda_{\mathcal{I}+}\}$: the one-loop dilation operator in these sectors is the same as in $\mathcal{N} = 4$ SYM.

Experimental tests of integrability will involve looking for degenerate “parity pairs” in the spectrum, as in [22, 12]. The ultimate proof of one-loop integrability would be to find an algebraic Bethe ansatz. In $\mathcal{N} = 4$ SYM, the universal R-matrix of the $SU(1, 1)$ subsector uplifts to the $PSU(2, 2|4)$ invariant R-matrix of the full theory [1]. In our case, the search for a candidate R-matrix should start in the $SU(1, 1) \times SU(1|1) \times SU(1|1)$ subsector. Work is in progress along these lines.

Another very interesting model that can be studied by our methods is $\mathcal{N} = 1$ SQCD at the upper edge of the conformal window ($N_f \sim 3N_c$). This theory has a large N Banks-Zaks fixed point and can be studied in perturbation theory. Its planar one-loop Hamiltonian in the scalar sector has been recently evaluated in [23], and shown to coincide with the Ising model in transverse magnetic field, which is of course integrable. This however may be a coincidence due to the simplicity of the scalar sector and it is important to look at the structure of the full theory. We have identified a closed $SU(1, 1) \times SU(1|1)$ subsector from which the full spin-chain Hamiltonian of $\mathcal{N} = 1$ SQCD can be uplifted. It will be interesting to see whether $\mathcal{N} = 1$ superconformal symmetry is in fact fixing the answer uniquely, and whether integrability extends to the full Hamiltonian.

Irrespective of integrability, the interpolating quiver theory and its string dual are a rich theoretical playground. They have been explored from a variety of viewpoints [8, 9, 10, 19, 24]. While integrability is broken away from the orbifold point, one retains remarkable analytic control, and our results are another indication of the intrinsic simplicity of this model.

Acknowledgements

It is a pleasure to thank Niklas Beisert, Carlo Meneghelli, Vladimir Mitev, Jan Plefka, Christoph Sieg, Matthias Staudacher and George Sterman for useful discussions and correspondence. E.P. wishes to thank the IHES for its warm hospitality as this work was in

progress. The work of P.L. and L.R. was supported in part by DOE grant DEFG-0292-ER40697 and by NSF grant PHY-0653351-001. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The work of E.P. is supported in part by the Humboldt Foundation.

A. $\mathcal{N} = 2$ Superconformal Multiplets

Detailed studies of the possible shortening conditions for the $\mathcal{N} = 2$ superconformal algebra were performed in [25, 26, 20]. In this appendix we summarize their findings in Table 3, following the conventions of [20].

Shortening Conditions				Multiplet
\mathcal{B}_1	$\mathcal{Q}_\alpha^1 R, r\rangle^{h.w.} = 0$	$j = 0$	$\Delta = 2R + r$	$\mathcal{B}_{R,r(0,\bar{j})}$
$\bar{\mathcal{B}}_2$	$\bar{\mathcal{Q}}_{2\dot{\alpha}} R, r\rangle^{h.w.} = 0$	$\bar{j} = 0$	$\Delta = 2R - r$	$\bar{\mathcal{B}}_{R,r(j,0)}$
\mathcal{E}	$\mathcal{B}_1 \cap \mathcal{B}_2$	$R = 0$	$\Delta = r$	$\mathcal{E}_{r(0,\bar{j})}$
$\bar{\mathcal{E}}$	$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$R = 0$	$\Delta = -r$	$\bar{\mathcal{E}}_{r(j,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1 \cap \bar{\mathcal{B}}_2$	$r = 0, j, \bar{j} = 0$	$\Delta = 2R$	$\hat{\mathcal{B}}_R$
\mathcal{C}_1	$\epsilon^{\alpha\beta} \mathcal{Q}_\beta^1 R, r\rangle_\alpha^{h.w.} = 0$		$\Delta = 2 + 2j + 2R + r$	$\mathcal{C}_{R,r(j,\bar{j})}$
	$(\mathcal{Q}^1)^2 R, r\rangle^{h.w.} = 0$ for $j = 0$		$\Delta = 2 + 2R + r$	$\mathcal{C}_{R,r(0,\bar{j})}$
$\bar{\mathcal{C}}_2$	$\epsilon^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{Q}}_{2\dot{\beta}} R, r\rangle_{\dot{\alpha}}^{h.w.} = 0$		$\Delta = 2 + 2\bar{j} + 2R - r$	$\bar{\mathcal{C}}_{R,r(j,\bar{j})}$
	$(\bar{\mathcal{Q}}_2)^2 R, r\rangle^{h.w.} = 0$ for $\bar{j} = 0$		$\Delta = 2 + 2R - r$	$\bar{\mathcal{C}}_{R,r(j,0)}$
\mathcal{F}	$\mathcal{C}_1 \cap \mathcal{C}_2$	$R = 0$	$\Delta = 2 + 2j + r$	$\mathcal{C}_{0,r(j,\bar{j})}$
$\bar{\mathcal{F}}$	$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R = 0$	$\Delta = 2 + 2\bar{j} - r$	$\bar{\mathcal{C}}_{0,r(j,\bar{j})}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1 \cap \bar{\mathcal{C}}_2$	$r = \bar{j} - j$	$\Delta = 2 + 2R + j + \bar{j}$	$\hat{\mathcal{C}}_{R(j,\bar{j})}$
$\hat{\mathcal{F}}$	$\mathcal{C}_1 \cap \mathcal{C}_2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R = 0, r = \bar{j} - j$	$\Delta = 2 + j + \bar{j}$	$\hat{\mathcal{C}}_{0(j,\bar{j})}$
\mathcal{D}	$\mathcal{B}_1 \cap \bar{\mathcal{C}}_2$	$r = \bar{j} + 1$	$\Delta = 1 + 2R + \bar{j}$	$\mathcal{D}_{R(0,\bar{j})}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}}_2 \cap \mathcal{C}_1$	$-r = j + 1$	$\Delta = 1 + 2R + j$	$\bar{\mathcal{D}}_{R(j,0)}$
\mathcal{G}	$\mathcal{E} \cap \bar{\mathcal{C}}_2$	$r = \bar{j} + 1, R = 0$	$\Delta = r = 1 + \bar{j}$	$\mathcal{D}_{0(0,\bar{j})}$
$\bar{\mathcal{G}}$	$\bar{\mathcal{E}} \cap \mathcal{C}_1$	$-r = j + 1, R = 0$	$\Delta = -r = 1 + j$	$\bar{\mathcal{D}}_{0(j,0)}$

Table 3: Shortening conditions and short multiplets for the $\mathcal{N} = 2$ superconformal algebra.

A generic long multiplet of the $\mathcal{N} = 2$ superconformal algebra is denoted by $\mathcal{A}_{R,r(j,\bar{j})}^\Delta$. It is generated by the action of the 8 Poincaré supercharges \mathcal{Q} and $\bar{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all the conformal supercharges \mathcal{S} . When some combination of the \mathcal{Q} 's also annihilates the primary, the corresponding multiplet is shorter. $|R, r\rangle_{(j,\bar{j})}^{h.w.}$ is the highest weight state with eigenvalues (R, r, j, \bar{j}) under the Cartan generators of the $SU(2)_R \times U(1)_r$ R-symmetry and of the Lorentz group. The multiplet built on this state is denoted as $\mathcal{X}_{R,r(j,\bar{j})}$, where the letter \mathcal{X} characterizes the shortening condition. The left column of Table 3 labels the condition. A superscript on the label corresponds to the index

$\mathcal{I} = 1, 2$ of the supercharge that kills the primary: for example \mathcal{B}_1 refers to \mathcal{Q}_α^1 . Similarly a “bar” on the label refers to the conjugate condition: for example $\bar{\mathcal{B}}_2$ corresponds to $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$ annihilating the state; this would result in the short anti-chiral multiplet $\bar{\mathcal{B}}_{R,r(j,0)}$, obeying $\Delta = 2R - r$. Note that conjugation reverses the signs of r , j and \bar{j} in the expression of the conformal dimension.

B. Oscillator Representation

In this appendix we describe the oscillator representation of the $\mathcal{N} = 2$ superconformal algebra $SU(2, 2|2)$. We introduce two sets of bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}_\alpha^\dagger)$, $(\mathbf{b}^{\dot{\alpha}}, \mathbf{b}_{\dot{\alpha}}^\dagger)$ and one set of fermionic oscillators $(\mathbf{c}^\mathcal{I}, \mathbf{c}_\mathcal{I}^\dagger)$, where $(\alpha, \dot{\alpha})$ are Lorentz indices and \mathcal{I} is an $SU(2)_R$ index. In addition we will need two more “auxiliary” fermionic operators $(\mathbf{d}, \mathbf{d}^\dagger)$ and $(\tilde{\mathbf{d}}, \tilde{\mathbf{d}}^\dagger)$. The non-zero (anti)commutation relations are

$$[\mathbf{a}^\alpha, \mathbf{a}_\beta^\dagger] = \delta_\beta^\alpha, \quad (\text{B.1})$$

$$[\mathbf{b}^{\dot{\alpha}}, \mathbf{b}_{\dot{\beta}}^\dagger] = \delta_{\dot{\beta}}^{\dot{\alpha}}, \quad (\text{B.2})$$

$$\{\mathbf{c}^\mathcal{I}, \mathbf{c}_\mathcal{J}^\dagger\} = \delta_\mathcal{J}^\mathcal{I}, \quad (\text{B.3})$$

$$\{\mathbf{d}, \mathbf{d}^\dagger\} = \{\tilde{\mathbf{d}}, \tilde{\mathbf{d}}^\dagger\} = 1. \quad (\text{B.4})$$

In this oscillator representation the generators of $SU(2, 2|2)$ read

$$\mathcal{Q}_\alpha^\mathcal{I} = \mathbf{a}_\alpha^\dagger \mathbf{c}^\mathcal{I}, \quad (\text{B.5})$$

$$\mathcal{S}_\mathcal{I}^\alpha = \mathbf{c}_\mathcal{I}^\dagger \mathbf{a}^\alpha, \quad (\text{B.6})$$

$$\tilde{\mathcal{Q}}_{\dot{\alpha}\mathcal{I}} = \mathbf{b}_{\dot{\alpha}}^\dagger \mathbf{c}_\mathcal{I}^\dagger, \quad (\text{B.7})$$

$$\tilde{\mathcal{S}}^{\dot{\alpha}\mathcal{I}} = \mathbf{b}^{\dot{\alpha}} \mathbf{c}^\mathcal{I}, \quad (\text{B.8})$$

$$\mathcal{P}_{\alpha\dot{\beta}} = \mathbf{a}_\alpha^\dagger \mathbf{b}_{\dot{\beta}}^\dagger, \quad (\text{B.9})$$

$$\mathcal{K}^{\alpha\dot{\beta}} = \mathbf{a}^\alpha \mathbf{b}^{\dot{\beta}}, \quad (\text{B.10})$$

$$\mathcal{L}_\beta^\alpha = \mathbf{a}_\beta^\dagger \mathbf{a}^\alpha - \frac{1}{2} \delta_\beta^\alpha \mathbf{a}_\gamma^\dagger \mathbf{a}^\gamma, \quad (\text{B.11})$$

$$\dot{\mathcal{L}}_{\dot{\beta}}^{\dot{\alpha}} = \mathbf{b}_{\dot{\beta}}^\dagger \mathbf{b}^{\dot{\alpha}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} \mathbf{b}_\gamma^\dagger \mathbf{b}^\gamma, \quad (\text{B.12})$$

$$\mathcal{R}_\mathcal{J}^\mathcal{I} = \mathbf{c}_\mathcal{J}^\dagger \mathbf{c}^\mathcal{I} - \frac{1}{2} \delta_\mathcal{J}^\mathcal{I} \mathbf{c}_\mathcal{K}^\dagger \mathbf{c}^\mathcal{K}, \quad (\text{B.13})$$

$$r = -\frac{1}{2} \mathbf{c}_\mathcal{K}^\dagger \mathbf{c}^\mathcal{K} + \frac{1}{2} \mathbf{d}^\dagger \mathbf{d} + \frac{1}{2} \tilde{\mathbf{d}}^\dagger \tilde{\mathbf{d}}, \quad (\text{B.14})$$

$$D = 1 + \frac{1}{2} \mathbf{a}_\gamma^\dagger \mathbf{a}^\gamma + \frac{1}{2} \mathbf{b}_\gamma^\dagger \mathbf{b}^\gamma, \quad (\text{B.15})$$

$$C = 1 - \frac{1}{2} \mathbf{a}_\gamma^\dagger \mathbf{a}^\gamma + \frac{1}{2} \mathbf{b}_\gamma^\dagger \mathbf{b}^\gamma - \frac{1}{2} \mathbf{c}_\mathcal{K}^\dagger \mathbf{c}^\mathcal{K} - \frac{1}{2} \mathbf{d}^\dagger \mathbf{d} - \frac{1}{2} \tilde{\mathbf{d}}^\dagger \tilde{\mathbf{d}}. \quad (\text{B.16})$$

Here C is a central charge that must kill any physical state. It could be eliminated from the

algebra by redefining $r + C \rightarrow r$, but it is useful for implementing the harmonic action so we will keep it. The quadratic Casimir operator is

$$J^2 = \frac{1}{2}D^2 + \frac{1}{2}\mathcal{L}_\alpha^\beta \mathcal{L}_\beta^\alpha + \frac{1}{2}\dot{\mathcal{L}}_{\dot{\alpha}}^{\dot{\beta}} \dot{\mathcal{L}}_{\dot{\beta}}^{\dot{\alpha}} - \frac{1}{2}\mathcal{R}_{\mathcal{I}}^{\mathcal{J}} \mathcal{R}_{\mathcal{J}}^{\mathcal{I}} - \frac{1}{2}[\mathcal{Q}_\alpha^{\mathcal{I}}, \mathcal{S}_{\mathcal{I}}^\alpha] - \frac{1}{2}[\tilde{\mathcal{Q}}_{\dot{\alpha}\mathcal{I}}, \tilde{\mathcal{S}}^{\dot{\alpha}\mathcal{I}}] - \frac{1}{2}\{\mathcal{P}_{\alpha\dot{\beta}}, \mathcal{K}^{\alpha\dot{\beta}}\} - \frac{1}{2}(r + C)(r + C). \quad (\text{B.17})$$

B.1 Vector multiplets \mathcal{V} and $\bar{\mathcal{V}}$

We define a vacuum state $|0\rangle$ annihilated by all the lowering operators. Then we identify

$$\mathcal{D}^k \mathcal{F} = (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k (\mathbf{c}^\dagger)^0 |0\rangle, \quad (\text{B.18})$$

$$\mathcal{D}^k \lambda = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k (\mathbf{c}^\dagger)^1 |0\rangle, \quad (\text{B.19})$$

$$\mathcal{D}^k \phi = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^k (\mathbf{c}^\dagger)^2 |0\rangle, \quad (\text{B.20})$$

and

$$\mathcal{D}^k \bar{\mathcal{F}} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+2} (\mathbf{c}^\dagger)^2 \tilde{\mathbf{d}}^\dagger \tilde{\mathbf{d}}^\dagger |0\rangle, \quad (\text{B.21})$$

$$\mathcal{D}^k \bar{\lambda} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+1} (\mathbf{c}^\dagger)^1 \tilde{\mathbf{d}}^\dagger \tilde{\mathbf{d}}^\dagger |0\rangle, \quad (\text{B.22})$$

$$\mathcal{D}^k \bar{\phi} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^k (\mathbf{c}^\dagger)^0 \tilde{\mathbf{d}}^\dagger \tilde{\mathbf{d}}^\dagger |0\rangle. \quad (\text{B.23})$$

For example,

$$\lambda_{\mathcal{I}\alpha} = \mathbf{a}_\alpha^\dagger \mathbf{c}_{\mathcal{I}}^\dagger |0\rangle, \quad \bar{\lambda}_{\mathcal{I}\dot{\alpha}} = \mathbf{b}_{\dot{\alpha}}^\dagger \mathbf{c}_{\mathcal{I}}^\dagger \tilde{\mathbf{d}}^\dagger |0\rangle. \quad (\text{B.24})$$

It's easy to see that all the quantum numbers match, including the zero central charge constraint.

B.2 Hypermultiplet \mathcal{H}

Similarly, for the hypermultiplet we identify

$$\mathcal{D}^k Q = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^k (\mathbf{c}^\dagger)^1 \mathbf{d}^\dagger |0\rangle, \quad (\text{B.25})$$

$$\mathcal{D}^k \bar{Q} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^k (\mathbf{c}^\dagger)^1 \tilde{\mathbf{d}}^\dagger |0\rangle, \quad (\text{B.26})$$

$$\mathcal{D}^k \psi = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \mathbf{d}^\dagger |0\rangle, \quad (\text{B.27})$$

$$\mathcal{D}^k \tilde{\psi} = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \tilde{\mathbf{d}}^\dagger |0\rangle, \quad (\text{B.28})$$

$$\mathcal{D}^k \bar{\psi} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+1} (\mathbf{c}^\dagger)^2 \mathbf{d}^\dagger |0\rangle, \quad (\text{B.29})$$

$$\mathcal{D}^k \tilde{\bar{\psi}} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+1} (\mathbf{c}^\dagger)^2 \tilde{\mathbf{d}}^\dagger |0\rangle. \quad (\text{B.30})$$

B.3 Two-letter Superconformal Primaries

By demanding that they are annihilated by all the conformal supercharges and by the appropriate combinations of Poincaré supercharges, we have worked out the expressions for the superconformal primaries of the irreducible modules that appear on the right hand side of the

tensor products (3.3)–(3.8). The grassmannOps.m oscillator package by Jeremy Michelson and Matthew Headrick was extremely useful for this task. We simply quote the results:

$\mathcal{V} \times \mathcal{V}$:

$$\bar{\mathcal{E}}_{2(0,0)} = \phi\phi, \quad (\text{B.31})$$

$$\bar{\mathcal{D}}_{\frac{1}{2}(\frac{1}{2},0)} = \lambda_{1+}\phi - \phi\lambda_{1+}, \quad (\text{B.32})$$

$$\begin{aligned} \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} &= \sum_{k=0}^{q-1} \frac{(-1)^k}{k+1} \binom{q-1}{k} \binom{q}{k} \left(\mathcal{D}^{q-k-1}\lambda_{1+}\mathcal{D}^k\lambda_{2+} - \mathcal{D}^{q-k-1}\lambda_{2+}\mathcal{D}^k\lambda_{1+} \right) \\ &\quad + \frac{1}{q+1} \left(\sum_{k=0}^{q-1} (-1)^k \binom{q-1}{k} \binom{q+1}{k} \mathcal{D}^{q-k-1}\mathcal{F}_{++}\mathcal{D}^k\phi \right. \\ &\quad \left. + \sum_{k=0}^{q-1} (-1)^k \binom{q-1}{k} \binom{q+1}{k+2} \mathcal{D}^{q-k-1}\phi\mathcal{D}^k\mathcal{F}_{++} \right). \end{aligned} \quad (\text{B.33})$$

For $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$ the expressions are identical with $(\phi, \lambda, \mathcal{F})$ replaced by $(\bar{\phi}, \bar{\lambda}, \bar{\mathcal{F}})$. The Casimir operator acting on these modules gives

$$J_{12}^2 \bar{\mathcal{E}}_{2(0,0)} = 0, \quad (\text{B.34})$$

$$J_{12}^2 \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} = (q+1)(q+2)\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})}, \quad q \geq -1. \quad (\text{B.35})$$

$\mathcal{V} \times \mathcal{H}$:

$$\bar{\mathcal{D}}_{\frac{1}{2}(0,0)} = \phi Q_1, \quad (\text{B.36})$$

$$\begin{aligned} \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})} &= \sum_{k=0}^q (-1)^k \binom{q}{k} \binom{q+1}{k} \left(\mathcal{D}^{q-k}\lambda_{2+}\mathcal{D}^k Q_1 - \mathcal{D}^{q-k}\lambda_{1+}\mathcal{D}^k Q_2 \right) \\ &\quad - \sum_{k=0}^q (-1)^k \binom{q}{k} \binom{q+1}{k+1} \mathcal{D}^{q-k}\phi\mathcal{D}^k\psi_+ \\ &\quad + q \sum_{k=0}^{q-1} \frac{(-1)^k}{k+1} \binom{q-1}{k} \binom{q+1}{k} \mathcal{D}^{q-k-1}\mathcal{F}_{++}\mathcal{D}^k\bar{\psi}_+. \end{aligned} \quad (\text{B.37})$$

As before, for $\bar{\mathcal{V}} \times H$ we replace $(\phi, \lambda, \mathcal{F})$ and $(\psi, \bar{\psi})$ by its conjugates. The action of the Casimir is

$$J_{12}^2 \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})} = (q + \frac{3}{2})(q + \frac{5}{2})\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})}, \quad q \geq -1. \quad (\text{B.38})$$

$\mathcal{H} \times \mathcal{V}$:

$$\bar{\mathcal{D}}_{\frac{1}{2}(0,0)} = Q_1 \check{\phi}, \quad (\text{B.39})$$

$$\begin{aligned} \hat{\mathcal{C}}_{0(\frac{q+1}{2}, \frac{q}{2})} &= \sum_{k=0}^q (-1)^k \binom{q}{k} \binom{q+1}{k+1} \left(\mathcal{D}^{q-k} Q_2 \mathcal{D}^k \check{\lambda}_{1+} - \mathcal{D}^{q-k} Q_1 \mathcal{D}^k \check{\lambda}_{2+} \right) \\ &\quad + \sum_{k=0}^q (-1)^k \binom{q}{k} \binom{q+1}{k} \mathcal{D}^{q-k} \psi_+ \mathcal{D}^k \check{\phi} \\ &\quad + q \sum_{k=0}^{q-1} \frac{(-1)^k}{k+2} \binom{q-1}{k} \binom{q+1}{k+1} \mathcal{D}^{q-k-1} \bar{\psi}_{\dot{+}} \mathcal{D}^k \check{\mathcal{F}}_{++}. \end{aligned} \quad (\text{B.40})$$

$\mathcal{H} \times \mathcal{H}$:

$$\hat{\mathcal{B}}_1 = Q_1 \bar{Q}_1, \quad (\text{B.41})$$

$$\begin{aligned} \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})} &= \sum_{k=0}^q (-1)^k \binom{q}{k} \binom{q}{k} \left(\mathcal{D}^{q-k} Q_1 \mathcal{D}^k \bar{Q}_2 - \mathcal{D}^{q-k} Q_2 \mathcal{D}^k \bar{Q}_1 \right) \\ &\quad + q \sum_{k=0}^{q-1} \frac{(-1)^k}{k+1} \binom{q}{k} \binom{q+1}{k} \mathcal{D}^{q-k} \psi_+ \mathcal{D}^k \bar{\psi}_{\dot{+}} \\ &\quad - q \sum_{k=0}^{q-1} \frac{(-1)^k}{k+1} \binom{q}{k} \binom{q+1}{k} \mathcal{D}^{q-k} \bar{\psi}_{\dot{+}} \mathcal{D}^k \tilde{\psi}_+, \end{aligned} \quad (\text{B.42})$$

with

$$J_{12}^2 \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})} = (q+1)(q+2) \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})}, \quad q \geq -1. \quad (\text{B.43})$$

$\mathcal{V} \times \bar{\mathcal{V}}$:

$$\begin{aligned} \hat{\mathcal{C}}_{0(\frac{q}{2}, \frac{q}{2})} &= \sqrt{\frac{2(q+2)}{q+1}} \left(\sum_{k=0}^q (-1)^k \binom{q}{k} \binom{q}{k} \mathcal{D}^{q-k} \phi \mathcal{D}^k \bar{\phi} \right. \\ &\quad - q \sum_{k=0}^{q-1} \frac{(-1)^k}{k+1} \binom{q}{k} \binom{q+1}{k} \left(\mathcal{D}^{q-k} \lambda_{1+} \mathcal{D}^k \bar{\lambda}_{2\dot{+}} - \mathcal{D}^{q-k} \lambda_{2+} \mathcal{D}^k \bar{\lambda}_{1\dot{+}} \right) \\ &\quad \left. + q \sum_{k=0}^{q-2} \frac{(-1)^k}{k+2} \binom{q+1}{k+1} \binom{q+2}{k} \mathcal{D}^{q-k} \mathcal{F}_{++} \mathcal{D}^k \bar{\mathcal{F}}_{\dot{+}\dot{+}} \right). \end{aligned} \quad (\text{B.44})$$

For $\bar{\mathcal{V}} \times \mathcal{V}$ we conjugate all fields.

C. A Sample Field Theory Calculation

In this appendix we work out an example of a Feynman diagram calculation of the one-loop dilation operator. We consider the $H_{12} \lambda_k \bar{\lambda}_{n-k} \rightarrow \lambda_k \bar{\lambda}_{n-k}$ mixing matrix element. For this we

require finiteness of the correlation function,

$$\int d^4x_i e^{-ik_i x_i} \frac{1}{k!(n-k)!} \langle \mathcal{D}^k \lambda(x) \mathcal{D}^{n-k} \bar{\lambda}(x) \bar{\lambda}(x_1) \lambda(x_2) \rangle. \quad (\text{C.1})$$

The 1P1 Feynman diagrams are given in the second line of Figure 4. The first is a “gauge emission” diagram coming from one of the covariant derivatives acting on the field, the second is a standard gauge loop and the last one is a Yukawa loop that contributes to $\lambda_k \bar{\lambda}_{n-k} \rightarrow \bar{\lambda}_k \lambda_{n-k}$ but not to $\lambda_k \bar{\lambda}_{n-k} \rightarrow \lambda_k \bar{\lambda}_{n-k}$ so we ignore it. To these contributions we have to add one half of the self-energy (Figure 1) of each external leg to obtain the Hamiltonian of the spin chain (see *e.g.* Section 2 of [27] and Appendix B of [12] for more details). We regularize the divergent integrals using a momentum cut-off. The tree level diagram is

$$\frac{(-i)^n}{k!(n-k)!} \frac{k_1^{k+1}}{k_1^2} \frac{k_2^{n-k+1}}{k_2^2}, \quad (\text{C.2})$$

where k_1^{k+1} and k_2^{n-k+1} are shorthands for k_{1++}^{k+1} and k_{2++}^{n-k+1} . We will usually suppress the indices and the slash, the powers of k and $n-k$ should help avoid confusion. For example,

$$(-k_2 - p)^{n-k} \equiv (-k_{2++} - p_{++})^{n-k}. \quad (\text{C.3})$$

The contribution to the Hamiltonian is minus the coefficient of $\frac{g_{YM}^2 N_c}{8\pi^2} \ln \Lambda$ (taking into account the tree level normalization).

For the gauge loop the standard Feynman rules give (factoring out $ig_{YM}^2 N_c$)

$$\frac{(-1)}{k_1^2 k_2^2} \int \frac{d^4 p}{(2\pi)^4} (i(p - k_1))^k (i(-k_2 - p))^{n-k} \frac{[(p - k_1) \bar{\sigma}^\mu k_1]_{\alpha\beta} [(k_2) \bar{\sigma}^\nu (p + k_2)]_{\beta\dot{\alpha}} \Delta_{\mu\nu}(p)}{(p - k_1)^2 (p + k_2)^2}.$$

For $(\alpha, \dot{\alpha}) = (+, \dot{+})$ and $(\beta, \dot{\beta}) = (+, \dot{+})$ in Feynman gauge we obtain

$$2 \frac{(i)^n}{k_1^2 k_2^2} k_{1\gamma\dot{+}} k_{2+\dot{\delta}} \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon^{\gamma\delta} \sigma_{+\dot{\gamma}}^\lambda \sigma_{\delta\dot{+}}^\rho \int \frac{d^4 p}{(2\pi)^4} \frac{(p - k_1)^k (-k_2 - p)^{n-k}}{p^2 (p - k_1)^2 (p + k_2)^2} (p - k_1)_\lambda (p + k_2)_\rho. \quad (\text{C.4})$$

Let's concentrate on the integral, redefining $p \rightarrow -k_2 - p$ we get

$$\int \frac{d^4 p}{(2\pi)^4} \frac{(-k_1 - k_2 - p)^k p^{n-k}}{p^2 (p + k_2)^2 (p + k_1 + k_2)^2} (p + k_1 + k_2)_\lambda p_\rho. \quad (\text{C.5})$$

Introducing Feynman parameters and defining

$$l = p + k_2 x + (k_1 + k_2) y, \quad (\text{C.6})$$

$$A = (k_1 + k_2) y + k_2 x - (k_1 + k_2), \quad (\text{C.7})$$

$$B = (k_1 + k_2) y + k_2 x, \quad (\text{C.8})$$

we obtain

$$2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4 l}{(2\pi)^4} \frac{(A-l)^k (l-B)^{n-k}}{(l^2 - \Delta)^3} (l-A)_\lambda (l-B)_\rho, \quad (\text{C.9})$$

where Δ are leftovers that do not affect the divergent part. Now, from this integral four kinds of Lorentz structure can appear: $g_{\lambda\rho}$, $g_{\lambda,++}$, $g_{++,\rho}$ and $g_{++,++}$. Clearly $g_{++,++} \equiv 0$. It turns out that $g_{\lambda,++}$ and $g_{++,\rho}$ give also zero when contracted with $\sigma_{+\dot{\gamma}}^\lambda$ and $\sigma_{\delta+}^\rho$ respectively (see eq. (C.4)). The only contribution comes from $g_{\lambda\rho}$. Then,

$$2(-1)^{n-k} \int_0^1 dx \int_0^{1-x} dy (-k_1 - k_2 + k_2 x + (k_1 + k_2)y)^k (k_2 x + (k_1 + k_2)y)^{n-k} \frac{g_{\lambda\rho}}{4} i \frac{\ln \Lambda}{8\pi^2}.$$

Denoting the Feynman parameter integral by $I(n, k, k_1, k_2)$, the final result is

$$2 \frac{(-i)^{n+1}}{k_1^2 k_2^2} k_{1++} k_{2++} (-1)^k I(n, k, k_1, k_2) \frac{\ln \Lambda}{8\pi^2}. \quad (\text{C.10})$$

The integral $I(n, k, k_1, k_2)$ can be solved analytically,

$$I(n, k, k_1, k_2) = \frac{(-1)^k (n-k)! k!}{n+2} \sum_{k'=0}^n \left(\delta_{k=k'} + \delta_{k>k'} \frac{n-k+1}{n-k'+1} + \delta_{k<k'} \frac{k+1}{k'+1} \right) \frac{k_1^{k'}}{k'!} \frac{k_2^{n-k'}}{(n-k')!}. \quad (\text{C.11})$$

The contribution to the Hamiltonian is then

$$a_{n,k,k'} = -\frac{1}{n+2} \left(\delta_{k=k'} + \delta_{k>k'} \frac{n-k+1}{n-k'+1} + \delta_{k<k'} \frac{k+1}{k'+1} \right). \quad (\text{C.12})$$

The other diagrams can be calculated in a similar way, we list the results for completeness. The self-energy is

$$a_{n,k,k'} = \delta_{k=k'} (h(k+1) + h(n-k+1)), \quad (\text{C.13})$$

while the gauge “emission” diagram gives

$$a_{n,k,k'} = -\frac{\delta_{k \neq k'}}{|k-k'|} + \frac{\delta_{k>k'}}{n-k'+1} + \frac{\delta_{k<k'}}{k'+1}. \quad (\text{C.14})$$

The sum of the three contributions gives the result quoted in (4.31).

D. Two Closed Subsectors and the Magnon S-matrix

The scattering of magnons in the spin chain of the interpolating SCFT was studied in [9, 19]. The choice of the $\phi/\check{\phi}$ spin chain vacuum breaks the symmetry to $SU(2_\alpha) \times SU(2_{\check{I}}) \times SU(2_{\check{\alpha}}|2_I)$, see [19] for a detailed explanation. The scattering of two magnons is given by a factorized two-body S-matrix

$$S_{SU(2_\alpha) \times SU(2_{\check{I}}) \times SU(2_{\check{\alpha}}|2_I)} = S_{SU(2_\alpha) \times SU(2_{\check{I}})} \otimes S_{SU(2_{\check{\alpha}}|2_I)}. \quad (\text{D.1})$$

The $S_{SU(2_{\hat{\alpha}}|2_I)}$ S-matrix describes the scattering of magnons in the highest weight of $SU(2_{\alpha}) \times SU(2_{\hat{I}})$ and is fixed by symmetry to all loops (up to the overall phase), as a function of the single parameter κ [4, 19]. In this appendix we evaluate the one-loop approximation of $S_{SU(2_{\hat{\alpha}}|2_I)}$ for bifundamental magnons, using the explicit spin chain Hamiltonian, and find agreement with the algebraic analysis of [19]. For this task it is useful to consider a closed subsector, the *right* subsector

$$\left\{ \phi, \check{\phi}, \bar{\psi}_{\hat{\alpha}\hat{I}=\hat{1}}, \bar{\tilde{\psi}}_{\hat{\alpha}\hat{I}=\hat{1}}, Q_{I\hat{I}=\hat{1}}, \bar{Q}_{I\hat{I}=\hat{1}} \right\}. \quad (\text{D.2})$$

One can also evaluate the one-loop approximation to the other factor of the two-body S-matrix, $S_{SU(2_{\alpha}) \times SU(2_{\hat{I}})}$, which is not fixed by symmetry, by considering the *left* closed subsector

$$\left\{ \phi, \check{\phi}, \lambda_{I=1\alpha}, \check{\lambda}_{I=1\alpha}, Q_{I=1\hat{I}}, \bar{Q}_{I=1\hat{I}} \right\}. \quad (\text{D.3})$$

We have evaluated the Hamiltonian in both the *left* and *right* sector by direct Feynman diagrams calculations, finding perfect agreement with the results of sections 4 and 5.

Our results for both sectors are as follows:

$$H_{k,k+1} = \begin{matrix} & \phi\lambda & \lambda\phi & \check{\phi}\check{\lambda} & \check{\lambda}\check{\phi} & \lambda Q & Q\check{\lambda} & \bar{Q}\lambda & \check{\lambda}\bar{Q} & \lambda\lambda & \check{\lambda}\check{\lambda} \\ \begin{matrix} \phi\lambda \\ \lambda\phi \\ \check{\phi}\check{\lambda} \\ \check{\lambda}\check{\phi} \\ \lambda Q \\ Q\check{\lambda} \\ \bar{Q}\lambda \\ \check{\lambda}\bar{Q} \\ \lambda\lambda \\ \check{\lambda}\check{\lambda} \end{matrix} & \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\kappa^2 & -2\kappa^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\kappa^2 & 2\kappa^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\kappa & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\kappa & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 + 2\mathbb{K}_I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (4 + 2\mathbb{K}_I)\kappa^2 \end{pmatrix} \end{matrix}$$

$$H_{k,k+1} = \begin{matrix} & \phi\bar{\tilde{\psi}} & \bar{\tilde{\psi}}\check{\phi} & \bar{\tilde{\psi}}\phi & \check{\phi}\bar{\tilde{\psi}} & \bar{\tilde{\psi}}\bar{\tilde{\psi}} & \bar{\tilde{\psi}}\bar{\tilde{\psi}} \\ \begin{matrix} \phi\bar{\tilde{\psi}} \\ \bar{\tilde{\psi}}\check{\phi} \\ \bar{\tilde{\psi}}\phi \\ \check{\phi}\bar{\tilde{\psi}} \\ \psi\bar{\tilde{\psi}} \\ \bar{\tilde{\psi}}\psi \end{matrix} & \begin{pmatrix} \frac{3+\kappa^2}{2} & -2\kappa & 0 & 0 & 0 & 0 \\ -2\kappa & \frac{3\kappa^2+1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3+\kappa^2}{2} & -2\kappa & 0 & 0 \\ 0 & 0 & -2\kappa & \frac{3\kappa^2+1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + 3\kappa^2 + 2\kappa^2\mathbb{K}_I & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa^2 + 3 + 2\mathbb{K}_I \end{pmatrix} \end{matrix}$$

$$\oplus \begin{matrix} Q\bar{\psi} \\ \bar{\psi}\bar{Q} \\ \bar{\psi}Q \\ \bar{Q}\bar{\psi} \end{matrix} \begin{pmatrix} Q\bar{\psi} & \bar{\psi}\bar{Q} & \bar{\psi}Q & \bar{Q}\bar{\psi} \\ \frac{1}{2} + \frac{3}{2}\kappa^2 & -2\kappa^2 & 0 & 0 \\ -2\kappa^2 & \frac{1}{2} + \frac{3}{2}\kappa^2 & 0 & 0 \\ 0 & 0 & \frac{\kappa^2}{2} + \frac{3}{2} & -2 \\ 0 & 0 & -2 & \frac{\kappa^2}{2} + \frac{3}{2} \end{pmatrix}.$$

Here we have chosen the gauge parameter ξ such that the self-energy for $Q_{\mathcal{I}\hat{\mathcal{I}}}$ is zero,⁹ with this convention the above matrices can be used in conjunction with the scalar sector result of [9]. The trace operator in Lorentz space is $\mathbb{K}_l = \varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta}$ (using Wess-Bagger conventions). For example,

$$H_{12}\lambda_{\alpha 1}\lambda_{\beta 1} = 4\lambda_{\alpha 1}\lambda_{\beta 1} - 2\varepsilon_{\alpha\beta}\lambda_1^\gamma\lambda_{\gamma 1}. \quad (\text{D.5})$$

D.1 S-matrix in the Right Sector

We can now solve the two-body scattering problem in the *right* sector.

D.1.1 $\bar{\psi}\bar{\psi}$ and $\bar{\psi}\bar{\bar{\psi}}$ scattering

The index structure of the fields implies that there cannot be any transmission, $\bar{\bar{\psi}}$ must always be to the left of $\bar{\psi}$, the process is pure reflection. Our results for the four different combinations of fields and indices are summarized in the following table.

Incoming	Sector	Scattering Matrix
$\bar{\bar{\psi}}\bar{\psi}$	$1_{\dot{\alpha}} \otimes 3_L$	$S(p_1, p_2, \kappa)$
$\bar{\bar{\psi}}\bar{\bar{\psi}}$	$3_{\dot{\alpha}} \otimes 3_L$	-1
$\bar{\psi}\bar{\bar{\psi}}$	$1_{\dot{\alpha}} \otimes 3_L$	$S(p_1, p_2, 1/\kappa)$
$\bar{\psi}\bar{\psi}$	$3_{\dot{\alpha}} \otimes 3_L$	-1

Table 4: Components of the S-matrix in the *right* sector.

where

$$S(p_1, p_2, \kappa) = -\frac{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}}{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_2}}. \quad (\text{D.6})$$

D.1.2 $\bar{\psi}Q$, $Q\bar{\psi}$, $\bar{Q}\bar{\psi}$ and $\bar{\bar{\psi}}\bar{Q}$ scattering

These processes are a little bit more interesting because we can have reflection and transmission. Taking into account all four combinations we obtain where

⁹ $\xi = -1$ if we write the gauge propagator as

$$\Delta(k^2) = \frac{1}{k^2}(g_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}). \quad (\text{D.4})$$

Incoming	T and R matrices
$\bar{\psi}Q$	$T(p_1, p_2, \kappa), R(p_1, p_2, \kappa)$
$Q\bar{\psi}$	$T(p_1, p_2, 1/\kappa), R(p_1, p_2, 1/\kappa)$
$\bar{Q}\bar{\psi}$	$T(p_1, p_2, \kappa), R(p_1, p_2, \kappa)$
$\bar{\psi}\bar{Q}$	$T(p_1, p_2, 1/\kappa), R(p_1, p_2, 1/\kappa)$

Table 5: Transmission and reflection coefficients in the *right* sector.

$$T(p_1, p_2) = -\frac{1 - e^{-ip_2 + ip_1}}{\kappa e^{-ip_2} + \kappa e^{ip_1} - 2}, \quad (\text{D.7})$$

$$R(p_1, p_2) = -\frac{1 - \kappa e^{-ip_2} - \kappa e^{ip_1} + e^{-ip_2 + ip_1}}{\kappa e^{-ip_2} + \kappa e^{ip_1} - 2}. \quad (\text{D.8})$$

Comparison of Table 4 and Table 5 with equ.(3.12) of [19] shows perfect agreement.

D.2 S-matrix in the Left Sector.

Our results for $\lambda\lambda$ scattering are summarized in Table 6 below. We could not solve the λQ scattering problem analytically, but one may straightforwardly find numerical results if needed.

Incoming	Sector	Scattering Matrix
$\lambda\lambda$	$1_\alpha \otimes 3_R$	$S(p_1, p_2, \kappa = 1)$
$\lambda\lambda$	$3_\alpha \otimes 3_R$	-1

Table 6: Scattering coefficients in the *left* sector.

References

- [1] N. Beisert and M. Staudacher, *The $N=4$ SYM integrable super spin chain*, *Nucl.Phys.* **B670** (2003) 439–463, [[hep-th/0307042](#)].
- [2] M. Staudacher, *The Factorized S -matrix of CFT/AdS*, *JHEP* **0505** (2005) 054, [[hep-th/0412188](#)].
- [3] N. Beisert and M. Staudacher, *Long-range $psu(2,2|4)$ Bethe Ansatz for gauge theory and strings*, *Nucl.Phys.* **B727** (2005) 1–62, [[hep-th/0504190](#)]. In honor of Hans Bethe.
- [4] N. Beisert, *The $su(2|2)$ dynamic S -matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945, [[hep-th/0511082](#)].
- [5] N. Beisert, B. Eden, and M. Staudacher, *Transcendentality and Crossing*, *J.Stat.Mech.* **0701** (2007) P01021, [[hep-th/0610251](#)].
- [6] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $N=4$ Supersymmetric Yang-Mills Theory*, *Phys.Rev.Lett.* **103** (2009) 131601, [[arXiv:0901.3753](#)].
- [7] N. Beisert *et. al.*, *Review of AdS/CFT Integrability: An Overview*, [arXiv:1012.3982](#).
- [8] A. Gadde, E. Pomoni, and L. Rastelli, *The Veneziano Limit of $N = 2$ Superconformal QCD: Towards the String Dual of $N = 2$ $SU(N(c))$ SYM with $N(f) = 2 N(c)$* , [arXiv:0912.4918](#).
- [9] A. Gadde, E. Pomoni, and L. Rastelli, *Spin Chains in $N=2$ Superconformal Theories: From the Z_2 Quiver to Superconformal QCD*, [arXiv:1006.0015](#).
- [10] E. Pomoni and C. Sieg, *From $N=4$ gauge theory to $N=2$ conformal QCD: three-loop mixing of scalar composite operators*, [arXiv:1105.3487](#).
- [11] N. Beisert and R. Roiban, *The Bethe ansatz for $Z(S)$ orbifolds of $N = 4$ super Yang- Mills theory*, *JHEP* **11** (2005) 037, [[hep-th/0510209](#)].
- [12] N. Beisert, *The complete one loop dilatation operator of $N=4$ superYang-Mills theory*, *Nucl.Phys.* **B676** (2004) 3–42, [[hep-th/0307015](#)].
- [13] N. Beisert, *The Dilatation operator of $N=4$ super Yang-Mills theory and integrability*, *Phys.Rept.* **405** (2005) 1–202, [[hep-th/0407277](#)]. Ph.D. Thesis.
- [14] N. Beisert, *The $su(2|3)$ dynamic spin chain*, *Nucl. Phys.* **B682** (2004) 487–520, [[hep-th/0310252](#)].
- [15] B. I. Zwiebel, *$N=4$ SYM to two loops: Compact expressions for the non-compact symmetry algebra of the $su(1,1|2)$ sector*, *JHEP* **0602** (2006) 055, [[hep-th/0511109](#)].
- [16] B. I. Zwiebel, *Two-loop Integrability of Planar $N=6$ Superconformal Chern-Simons Theory*, *J.Phys.A* **A42** (2009) 495402, [[arXiv:0901.0411](#)].
- [17] S. Kachru and E. Silverstein, *4d conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855–4858, [[hep-th/9802183](#)].
- [18] I. R. Klebanov and N. A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, *Nucl. Phys.* **B574** (2000) 263–274, [[hep-th/9911096](#)].

- [19] A. Gadde and L. Rastelli, *Twisted Magnons*, [arXiv:1012.2097](#).
- [20] F. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41–89, [[hep-th/0209056](#)].
- [21] M. Bianchi, F. Dolan, P. Heslop, and H. Osborn, *$N=4$ superconformal characters and partition functions*, *Nucl.Phys.* **B767** (2007) 163–226, [[hep-th/0609179](#)].
- [22] N. Beisert, C. Kristjansen, and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B664** (2003) 131–184, [[hep-th/0303060](#)].
- [23] D. Poland and D. Simmons-Duffin, *$N=1$ SQCD and the Transverse Field Ising Model*, [arXiv:1104.1425](#).
- [24] S.-J. Rey and T. Suyama, *Exact Results and Holography of Wilson Loops in $N=2$ Superconformal (Quiver) Gauge Theories*, *JHEP* **01** (2011) 136, [[arXiv:1001.0016](#)].
- [25] V. Dobrev and V. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, *Phys.Lett.* **B162** (1985) 127–132.
- [26] V. Dobrev and V. Petkova, *GROUP THEORETICAL APPROACH TO EXTENDED CONFORMAL SUPERSYMMETRY: FUNCTION SPACE REALIZATIONS AND INVARIANT DIFFERENTIAL OPERATORS*, *Fortsch.Phys.* **35** (1987) 537.
- [27] J. Minahan and K. Zarembo, *The Bethe ansatz for $N=4$ superYang-Mills*, *JHEP* **0303** (2003) 013, [[hep-th/0212208](#)].